Glueing Booleans for Greater Extensionality

Kenji Maillard

Inria, team Gallinette

GdT LHC
Wednesday the 5th of June, 2024

A question from M. Shulman

Proof Assistants Beta

Home

PUBLIC

Questions

Tags

Users

Unanswered

TEAMS ×

Strong eta-rules for functions on sum types

Ask Question

Asked 2 months ago Modified 2 months ago Viewed 145 times

8

I am wondering whether a rule like the following is consistent with decidable conversion and type-checking for dependent type theory:

$$\frac{f\ g : (x : \mathsf{bool}) \to C\ x \qquad f\ \mathsf{tt} \equiv g\ \mathsf{tt} \qquad f\ \mathsf{ff} \equiv g\ \mathsf{ff}}{f \equiv g}$$

That is, if two functions with domain bool agree definitionally on tt and ff, then they are convertible. An analogous rule for functions on general inductive types like $\mathbb{N}$ is certainly

A question from M. Shulman

Proof Assistants Beta

Home

PUBLIC

Questions

Tags

Users

Unanswered

TEAMS ×

Strong eta-rules for functions on sum types

Ask Question

Asked 2 months ago Modified 2 months ago Viewed 145 times

8

I am wondering whether a rule like the following is consistent with decidable conversion and type-checking for dependent type theory:

$$\frac{f\ g : (x : \mathsf{bool}) \to C\ x \qquad f\ \mathsf{tt} \equiv g\ \mathsf{tt} \qquad f\ \mathsf{ff} \equiv g\ \mathsf{ff}}{f \equiv g}$$

That is, if two functions with domain bool agree definitionally on tt and ff, then they are convertible. An analogous rule for functions on general inductive types like $\mathbb{N}$ is certainly

Context: Martin-Löf Type Theories (with $\mathtt{Ty}, \Pi, \Sigma, x =_A y, \ldots$)

Extensional principles in intensional type theory

Type formers	Dec. of conv.	Reference
Functions $\Pi(x : A)B$	✓	[Coquand 96]
(Negative) records $\Sigma(x : A)B$	✓	[Norell 07]
Unit $\mathbb{1}$	✓	[Norell 07]
Identity $x =_A y$	×	[Castellan et al. 17]
Natural numbers $\mathbb{N}$	×	[RefNec]
Well-founded trees $\mathbb{W}(x : A)B$	×	Reduction to $\mathbb{N}$
Streams, $\mathbb{M}$-types	×	[McBride's Ripley]
Empty $\mathbb{0}$	×	[McBride's Ripley]
Booleans $\mathbb{B}$	???	

Booleans 101

Introductions

$$\frac{}{\Gamma \vdash \mathbb{B}} \qquad \frac{}{\Gamma \vdash \texttt{tt} : \mathbb{B}} \qquad \frac{}{\Gamma \vdash \texttt{ff} : \mathbb{B}}$$

Simple elimination

$$\frac{\Gamma \vdash b : \mathbb{B} \qquad \Gamma \vdash t : C \qquad \Gamma \vdash u : C}{\Gamma \vdash \texttt{if}\ b\ \texttt{then}\ t\ \texttt{else}\ u : C}$$

Booleans 101

Introductions

$$\frac{}{\Gamma \vdash \mathbb{B}} \qquad \frac{}{\Gamma \vdash \mathtt{tt} : \mathbb{B}} \qquad \frac{}{\Gamma \vdash \mathtt{ff} : \mathbb{B}}$$

Simple elimination

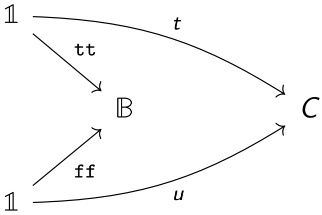
$$\frac{\Gamma \vdash b : \mathbb{B} \qquad \Gamma \vdash t : C \qquad \Gamma \vdash u : C}{\Gamma \vdash \mathtt{if}\ b\ \mathtt{then}\ t\ \mathtt{else}\ u : C}$$

Computation rules

$$\mathtt{if}\ \mathtt{tt}\ \mathtt{then}\ t\ \mathtt{else}\ u \longrightarrow t \qquad \mathtt{if}\ \mathtt{ff}\ \mathtt{then}\ t\ \mathtt{else}\ u \longrightarrow u$$

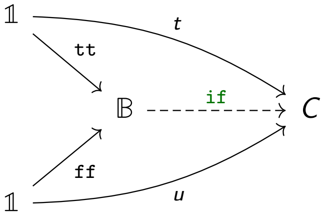
What's boolean extensionality ?

Returning to the categorical universal property of $\mathbb{B} \cong \mathbb{1} + \mathbb{1}$



What’s boolean extensionality ?

Returning to the categorical universal property of $\mathbb{B} \cong \mathbb{1} + \mathbb{1}$

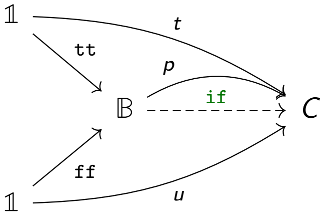


Existence part:

$$\texttt{if}(t, u) \circ \texttt{tt} \equiv t \qquad \texttt{if}(t, u) \circ \texttt{ff} \equiv u$$

What's boolean extensionality ?

Returning to the categorical universal property of $\mathbb{B} \cong \mathbb{1} + \mathbb{1}$



Existence part:

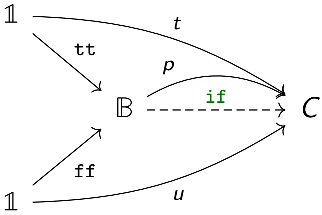
$$\texttt{if}(t, u) \circ \texttt{tt} \equiv t \qquad \texttt{if}(t, u) \circ \texttt{ff} \equiv u$$

Unicity part:

$$\frac{p : \mathbb{B} \to C}{}$$

What's boolean extensionality ?

Returning to the categorical universal property of $\mathbb{B} \cong \mathbb{1} + \mathbb{1}$



Existence part:

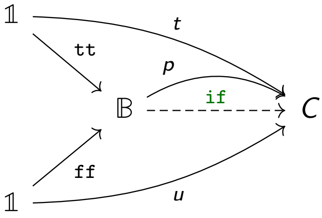
$$\mathtt{if}(t, u) \circ \mathtt{tt} \equiv t \qquad \mathtt{if}(t, u) \circ \mathtt{ff} \equiv u$$

Unicity part:

$$\frac{p : \mathbb{B} \to C \qquad p \circ \mathtt{tt} \equiv t \qquad p \circ \mathtt{ff} \equiv u}{}$$

What's boolean extensionality ?

Returning to the categorical universal property of $\mathbb{B} \cong \mathbb{1} + \mathbb{1}$



Existence part:

$$\mathtt{if}(t, u) \circ \mathtt{tt} \equiv t \qquad \mathtt{if}(t, u) \circ \mathtt{ff} \equiv u$$

Unicity part:

$$\frac{p : \mathbb{B} \to C \qquad p \circ \mathtt{tt} \equiv t \qquad p \circ \mathtt{ff} \equiv u}{p \equiv \mathtt{if}(t, u) : C}$$

What's boolean extensionality **in Type Theory** ?

Uniqueness categorically

$$\frac{p : \mathbb{B} \to C \qquad p \circ \mathtt{tt} \equiv t \qquad p \circ \mathtt{ff} \equiv u}{p \equiv \mathtt{if}(t, u) : C}$$

Uniqueness type-theoretically

$$\frac{\qquad}{\qquad}$$

What's boolean extensionality **in Type Theory** ?

Uniqueness categorically

$$\frac{p : \mathbb{B} \to C \qquad p \circ \mathtt{tt} \equiv t \qquad p \circ \mathtt{ff} \equiv u}{p \equiv \mathtt{if}(t, u) : C}$$

Uniqueness type-theoretically

$$\frac{\Gamma, b : \mathbb{B} \vdash p : C}{}$$

What's boolean extensionality **in Type Theory** ?

Uniqueness categorically

$$\frac{p : \mathbb{B} \to C \qquad p \circ \mathtt{tt} \equiv t \qquad p \circ \mathtt{ff} \equiv u}{p \equiv \mathtt{if}(t, u) : C}$$

Uniqueness type-theoretically

$$\frac{\Gamma, b : \mathbb{B} \vdash p : C \qquad \Gamma \vdash p[\mathtt{tt}/b] \equiv t : C}{}$$

What's boolean extensionality **in Type Theory** ?

Uniqueness categorically

$$\frac{p : \mathbb{B} \to C \qquad p \circ \texttt{tt} \equiv t \qquad p \circ \texttt{ff} \equiv u}{p \equiv \texttt{if}(t, u) : C}$$

Uniqueness type-theoretically

$$\frac{\Gamma, b : \mathbb{B} \vdash p : C \qquad \Gamma \vdash p[\texttt{tt}/b] \equiv t : C \qquad \Gamma \vdash p[\texttt{ff}/b] \equiv u : C}{}$$

What's boolean extensionality **in Type Theory** ?

Uniqueness categorically

$$\frac{p : \mathbb{B} \to C \qquad p \circ \texttt{tt} \equiv t \qquad p \circ \texttt{ff} \equiv u}{p \equiv \texttt{if}(t, u) : C}$$

Uniqueness type-theoretically

$$\frac{\Gamma, b : \mathbb{B} \vdash p : C \qquad \Gamma \vdash p[\texttt{tt}/b] \equiv t : C \qquad \Gamma \vdash p[\texttt{ff}/b] \equiv u : C}{\Gamma, b : \mathbb{B} \vdash p \equiv \texttt{if}\; b\; \texttt{then}\; t\; \texttt{else}\; u : C}$$

Booleans, Dependently

Dependent elimination

$$\frac{\Gamma \vdash b : \mathbb{B} \qquad \Gamma, x : \mathbb{B} \vdash P \qquad \Gamma \vdash t : P[\mathtt{tt}/x] \qquad \Gamma \vdash u : P[\mathtt{ff}/x]}{\Gamma \vdash \mathtt{ind}(x.P; b; t \mid u) : P[b/x]}$$

Booleans, Dependently

Dependent elimination

$$\frac{\Gamma \vdash b : \mathbb{B} \qquad \Gamma, x : \mathbb{B} \vdash P \qquad \Gamma \vdash t : P[\mathtt{tt}/x] \qquad \Gamma \vdash u : P[\mathtt{ff}/x]}{\Gamma \vdash \mathtt{ind}(x.P; b; t \mid u) : P[b/x]}$$

Computation rules

$$\mathtt{ind}(x.P; \mathtt{tt}; t \mid u) \longrightarrow t \qquad\qquad \mathtt{ind}(x.P; \mathtt{ff}; t \mid u) \longrightarrow u$$

Booleans, Dependently

Dependent elimination

$$\frac{\Gamma \vdash b : \mathbb{B} \qquad \Gamma, x : \mathbb{B} \vdash P \qquad \Gamma \vdash t : P[\texttt{tt}/x] \qquad \Gamma \vdash u : P[\texttt{ff}/x]}{\Gamma \vdash \texttt{ind}(x.P; b; t \mid u) : P[b/x]}$$

Computation rules

$$\texttt{ind}(x.P; \texttt{tt}; t \mid u) \longrightarrow t \qquad\qquad \texttt{ind}(x.P; \texttt{ff}; t \mid u) \longrightarrow u$$

Extensionality (Naively)

$$\frac{\Gamma, x : \mathbb{B} \vdash P \qquad \Gamma \vdash b : \mathbb{B} \qquad \Gamma \vdash p : P[b/x]}{}$$

Booleans, Dependently

Dependent elimination

$$\frac{\Gamma \vdash b : \mathbb{B} \qquad \Gamma, x : \mathbb{B} \vdash P \qquad \Gamma \vdash t : P[\texttt{tt}/x] \qquad \Gamma \vdash u : P[\texttt{ff}/x]}{\Gamma \vdash \texttt{ind}(x.P; b; t \mid u) : P[b/x]}$$

Computation rules

$$\texttt{ind}(x.P; \texttt{tt}; t \mid u) \longrightarrow t \qquad \texttt{ind}(x.P; \texttt{ff}; t \mid u) \longrightarrow u$$

Extensionality (Naively)

$$\frac{\Gamma \vdash p : P[b/x] \qquad \Gamma, x : \mathbb{B} \vdash P \qquad \Gamma \vdash b : \mathbb{B} \qquad \Gamma \vdash p[\texttt{tt}/b] \equiv t : P[\texttt{tt}/x] \qquad \Gamma \vdash p[\texttt{ff}/b] \equiv u : P[\texttt{ff}/x]}{}$$

Booleans, Dependently

Dependent elimination

$$\frac{\Gamma \vdash b : \mathbb{B} \qquad \Gamma, x : \mathbb{B} \vdash P \qquad \Gamma \vdash t : P[\mathtt{tt}/x] \qquad \Gamma \vdash u : P[\mathtt{ff}/x]}{\Gamma \vdash \mathtt{ind}(x.P; b; t \mid u) : P[b/x]}$$

Computation rules

$$\mathtt{ind}(x.P; \mathtt{tt}; t \mid u) \longrightarrow t \qquad \mathtt{ind}(x.P; \mathtt{ff}; t \mid u) \longrightarrow u$$

Extensionality (Naively)

$$\frac{\Gamma \vdash p : P[b/x] \qquad \Gamma, x : \mathbb{B} \vdash P \qquad \Gamma \vdash b : \mathbb{B} \qquad \Gamma \vdash p[\mathtt{tt}/b] \equiv t : P[\mathtt{tt}/x] \qquad \Gamma \vdash p[\mathtt{ff}/b] \equiv u : P[\mathtt{ff}/x]}{\Gamma \vdash p \equiv \mathtt{ind}(x.P; b; t \mid u) : P[b/x]}$$

With type dependency, substitution is not enough

Assuming $\alpha : \mathbb{N} \to \mathbb{B} \in \Gamma$, consider

$$\Gamma \vdash \mathtt{ind}(b.\forall n, \alpha\ n = b \to \mathbb{N}; \alpha\ 42; \lambda\ n\ eq.\ 0 \,|\, \lambda\ n\ eq.\ 0)\ 42\ \mathtt{refl} : \mathbb{N}$$

where $\quad \Gamma \vdash \mathtt{refl} : \alpha\ 42 = \alpha\ 42$

With type dependency, substitution is not enough

Assuming $\alpha : \mathbb{N} \to \mathbb{B} \in \Gamma$, consider

$$\Gamma \vdash \texttt{ind}(b.\forall n, \alpha\ n = b \to \mathbb{N}; \alpha\ 42; \lambda\ n\ eq.\ 0 \mid \lambda\ n\ eq.\ 0)\ 42\ \texttt{refl} : \mathbb{N}$$

$$\text{where} \qquad \Gamma \vdash \texttt{refl} : \alpha\ 42 = \alpha\ 42$$

But substituting α 42 by `tt` is ill-typed:

$$\Gamma \nvdash \texttt{ind}(b.\forall n, \alpha\ n = b \to \mathbb{N}; \texttt{tt}; \lambda\ n\ eq.\ 0 \mid \lambda\ n\ eq.\ 0)\ 42\ \texttt{refl} : \mathbb{N}$$

$$\text{where} \qquad \Gamma \nvdash \texttt{refl} : \alpha\ 42 = \texttt{tt}$$

Need to keep track of convertibility relations at $\mathbb{B}$!

Reflecting conversion at $\mathbb{B}$

Add boolean constraints [Altenkirch, Dybjer, Hoffman & Scott, 2001]

$$\frac{\Gamma \vdash \qquad \Gamma \vdash b : \mathbb{B} \qquad v \in \{\mathtt{tt}, \mathtt{ff}\}}{\Gamma, b \equiv v \vdash}$$

Reflecting conversion at $\mathbb{B}$

Add boolean constraints [Altenkirch,Dybjer,Hoffman & Scott, 2001]

$$\frac{\Gamma \vdash \qquad \Gamma \vdash b : \mathbb{B} \qquad v \in \{\texttt{tt}, \texttt{ff}\}}{\Gamma, b \equiv v \vdash}$$

Extend conversion

$$\textsc{Reflection}\ \frac{(b \equiv v) \in \Gamma}{\Gamma \vdash b \equiv v : \mathbb{B}} \qquad \textsc{Explosion}\ \frac{(b \equiv \texttt{tt}), (b \equiv \texttt{ff}) \in \Gamma \qquad \Gamma \vdash t, u : C}{\Gamma \vdash t \equiv u : C}$$

$$\textsc{Cover}\ \frac{\Gamma \vdash b : \mathbb{B} \qquad \Gamma, b \equiv \texttt{tt} \vdash t \equiv u : C \qquad \Gamma, b \equiv \texttt{ff} \vdash t \equiv u : C}{\Gamma \vdash t \equiv u : C}$$

Decidability of Conversion ?

Conversion checking may require full normal forms, e.g.

$$f : \mathbb{B} \to \mathbb{B} \vdash f \circ f \circ f \equiv f : \mathbb{B} \to \mathbb{B}$$

Decidability of Conversion ?

Conversion checking may require full normal forms, e.g.

$$f : \mathbb{B} \to \mathbb{B} \vdash f \circ f \circ f \equiv f : \mathbb{B} \to \mathbb{B}$$

Neutral boolean terms cannot be normal: normal forms need some intrinsic typing.

$\rightsquigarrow$ Manageable with simple types, hard with dependency

Decidability of Conversion ?

Conversion checking may require full normal forms, e.g.

$$f : \mathbb{B} \to \mathbb{B} \vdash f \circ f \circ f \equiv f : \mathbb{B} \to \mathbb{B}$$

Neutral boolean terms cannot be normal: normal forms need some intrinsic typing.
$\rightsquigarrow$ Manageable with simple types, hard with dependency

Solution: Kripke logical relations wrt renamings for (full) normalisation

- Normal/neutral forms are only stable by renamings, not arbitrary substitutions
- Collapse/Cover rules correspond to sheaf conditions

$$\varnothing \triangleright \Gamma, b \equiv \texttt{tt}, b \equiv \texttt{ff} \qquad \{(\Gamma, b \equiv \texttt{tt}), (\Gamma, b \equiv \texttt{ff})\} \triangleright \Gamma$$

- Hard to formalize explicitly already for simpler systems (e.g. MLTT with $\Pi, \texttt{Ty}$)

Roadmap to Normalization

Apply **internal sconing** [Bocquet, Kaposi & Sattler, 2023]

"**Synthetic** normalization proof for $\mathcal{T}$ internal to Psh($\mathcal{R}$en($\mathcal{T}$))"

Roadmap to Normalization

Apply **internal sconing** [Bocquet, Kaposi & Sattler, 2023]

"**Synthetic** normalization proof for $\mathcal{T}$ internal to $\mathsf{Psh}(\mathcal{R}\mathsf{en}(\mathcal{T}))$"

- Any (model of a) type theory $\mathcal{T}$ induces a category of renamings
- $\mathsf{Psh}(\mathcal{R}\mathsf{en}(\mathcal{T}))$ host an incarnation of the initial model of $\mathcal{T}$
- A proof method: relative induction principle with respect to renamings
- Internal sconing provides the elimination scheme for relative induction

Generalized Algebraic Theories

Playing with algebraic theories:

- Start from a second-order GAT $\Pi\mathbb{B}^{\text{ext}}_{\textbf{SOGAT}}$,
- Generate its first-ordification $\Pi\mathbb{B}^{\text{ext}}_{\textbf{GAT}}$

Juggling between models of

- a GAT internal to any Category with Family (CwF);
- a SOGAT inside any presheaf categories $\text{Psh}(C)$.

Any presheaf category is a Cwf, in particular $\mathcal{S}$et is a Cwf.

A SOGAT $\Pi\mathbb{B}^{\text{ext}}_{\text{SOGAT}}$ for Extensional Booleans

$$\mathsf{ty} : \mathbf{Sort} \qquad \mathsf{Bool} : \mathsf{ty}$$

$$\mathsf{tm} : \mathsf{ty} \to \mathbf{Sort} \qquad \mathsf{force} : (b \colon \mathsf{tm}\,\mathsf{Bool})(x \colon \mathbb{B}) \to \mathbf{Sort}$$

$$\mathsf{true}, \mathsf{false} : \mathsf{tm}\,\mathsf{Bool} \qquad \mathsf{If} : (b \colon \mathsf{tm}\,\mathsf{Bool})(A \colon \mathsf{force}\,b\,\top \to \mathsf{ty})(B \colon \mathsf{force}\,b\,\bot \to \mathsf{ty}) \to \mathsf{ty}$$

$$\mathsf{force/irr} : (b \colon \mathsf{tm}\,\mathsf{Bool})(x \colon \mathbb{B})(t\ u \colon \mathsf{force}\,b\,x) \to t \cong u$$

$$\mathsf{force/collapse/ty} : (b \colon \mathsf{tm}\,\mathsf{Bool})(A\ B \colon \mathsf{ty})(h^{\top} \colon \mathsf{force}\,b\,\top)(h^{\bot} \colon \mathsf{force}\,b\,\bot) \to A \cong B$$

$$\mathsf{force/collapse/tm} : (b \colon \mathsf{tm}\,\mathsf{Bool})(A \colon \mathsf{ty})(t\ u \colon \mathsf{tm}\,A)(h^{\top} \colon \mathsf{force}\,b\,\top)(h^{\bot} \colon \mathsf{force}\,b\,\bot) \to t \cong u$$

$$\mathsf{force/true} : \mathsf{force}\,\mathsf{true}\,\top \qquad \mathsf{force/conv/true} : (b \colon \mathsf{tm}\,\mathsf{Bool})(h \colon \mathsf{force}\,b\,\top) \to b \cong \mathsf{true}$$

$$\mathsf{force/false} : \mathsf{force}\,\mathsf{false}\,\bot \qquad \mathsf{force/conv/false} : (b \colon \mathsf{tm}\,\mathsf{Bool})(h \colon \mathsf{force}\,b\,\bot) \to b \cong \mathsf{false}$$

$$\mathsf{If/true} : (A \colon \mathsf{force}\,\mathsf{true}\,\top \to \mathsf{ty})(B \colon \mathsf{force}\,\mathsf{true}\,\bot \to \mathsf{ty}) \to \mathsf{If}\,\mathsf{true}\,A\,B \cong A\,\mathsf{force/true}$$

$$\mathsf{If/false} : (A \colon \mathsf{force}\,\mathsf{false}\,\top \to \mathsf{ty})(B \colon \mathsf{force}\,\mathsf{false}\,\bot \to \mathsf{ty}) \to \mathsf{If}\,\mathsf{false}\,A\,B \cong B\,\mathsf{force/false}$$

$$\mathsf{If/eta} : (b \colon \mathsf{tm}\,\mathsf{Bool})(A \colon \mathsf{ty}) \to A \cong \mathsf{If}\,b\,(\lambda h.A)\,(\lambda h.A)$$

First-ordification $\Pi\mathbb{B}^{\mathsf{ext}}_{\mathbf{GAT}}$ for Extensional Booleans

$$\mathrm{ctx} : \mathbf{Sort} \qquad \mathrm{sub} : \mathrm{ctx} \to \mathrm{ctx} \to \mathbf{Sort}$$

$$\mathrm{ty} : (\Gamma : \mathrm{ctx}) \to \mathbf{Sort} \qquad \mathrm{Bool} : \{\Gamma : \mathrm{ctx}\} \to \mathrm{ty}$$

$$\mathrm{tm} : (\Gamma : \mathrm{ctx})(A : \mathrm{ty}\,\Gamma) \to \mathbf{Sort} \qquad \mathrm{force} : (\Gamma : \mathrm{ctx})(b : \mathrm{tm}\,\Gamma\,\mathrm{Bool})(x : \mathbb{B}) \to \mathbf{Sort}$$

$$\varepsilon : \mathrm{ctx}$$

$$- \,.\, - : (\Gamma : \mathrm{ctx})(A : \mathrm{ty}\,\Gamma) \to \mathrm{ctx} \qquad -,\ - \overset{\mathrm{def}}{=} - : (\Gamma : \mathrm{ctx})(b : \mathrm{tm}\,\Gamma\,\mathrm{Bool})(x : \mathbb{B}) \to \mathrm{ctx}$$

$$\mathrm{substTy} : \Gamma\ \Delta : \mathrm{ctx}(\sigma : \mathrm{sub}\,\Gamma\,\Delta)(A : \mathrm{ty}\,\Delta) \to \mathrm{ty}\,\Gamma$$

$$\mathrm{If} : (\Gamma : \mathrm{ctx})(b : \mathrm{tm}\,\Gamma\,\mathrm{Bool})(A : \mathrm{ty}\,(\Gamma,\ b \overset{\mathrm{def}}{=} \top))(B : \mathrm{ty}\,(\Gamma,\ b \overset{\mathrm{def}}{=} \bot)) \to \mathrm{ty}\,\Gamma$$

Turning Coq into Psh($\mathcal{R}$en($\Pi\mathbb{B}^{\text{ext}}_{\textbf{GAT}}$))

- Based on the `coq-observational` branch of L. Pujet implementing ObsTT
- Axiomatize the structure specific to $\mathcal{R}$en($\Pi\mathbb{B}^{\text{ext}}_{\textbf{GAT}}$)

```
Axiom Ty : Set.
Symbol Tm : Ty -> Set.

Symbol Var : Ty -> Set.
Symbol var : forall (A : Ty), Var A -> Tm A.

Symbol boolty : Ty.
Symbol cstbool : bool -> Tm boolty.
Definition force (b : Tm boolty) c := b ~ cstbool c.

Symbol ifty : forall (b : Tm boolty) (A : force b true -> Ty) (B : force b false -> Ty), Ty.
Rewrite Rule ifty_red :=
| ifty (cstbool true) ?A ?B >-> ?A (obseq_refl _)
| ifty (cstbool false) ?A ?B >-> ?B (obseq_refl _).
```

Synthetic Normalisation inside Coq

Manipulating sheaves following P.M. Pédrot

- Right orthogonality wrt to a family of strict propositions.
- Sheafification as a quotient-inductive type.

Synthetic Normalisation inside Coq

Manipulating sheaves following P.M. Pédrot

- Right orthogonality wrt to a family of strict propositions.
- Sheafification as a quotient-inductive type.

Define intrinsically typed normal/neutral forms

- with quotient functions to terms/types,
- inducing a new topology based on booleans neutrals,
- and producing sheaves for that topology.
- An instance of a quotient inductive recursive type.

Synthetic Normalisation inside Coq

Manipulating sheaves following P.M. Pédrot

- Right orthogonality wrt to a family of strict propositions.
- Sheafification as a quotient-inductive type.

Define intrinsically typed normal/neutral forms

- with quotient functions to terms/types,
- inducing a new topology based on booleans neutrals,
- and producing sheaves for that topology.
- An instance of a quotient inductive recursive type.

```
(* Normalization structure *)
Record norm {A : Ty} := {
  normPred : Tm A -> shU ;
  quote : forall (a : Tm A), shEl (normPred a) -> fib nfToTm a ;
  reflect : forall (a : ne A), shEl (normPred (neToTm a))
}.
```

Conclusion

Current state:

- WIP synthetic proof of normalization for extensional booleans in Coq
- Rely on internal sconing and relative induction with respect to renamings
- Decidability of conversion through an explicit construction of sheafification

Future steps:

- What's ultimately assumed about the metatheory ? Choice principles for QITs ?
- Can we validate the axioms inside Coq using a prefascist translation ?
- Ongoing work on a toy normalization-by-evaluation typechecker in ocaml
- Correctness of the typechecker ?

Renamings with Forced Booleans

Assuming a model $\mathcal{M}$ of $\Pi\mathbb{B}^{\text{ext}}_{\textbf{GAT}}$, define $\mathcal{R}\text{en}(\mathcal{M})$

$$\varepsilon \in \mathrm{Obj}(\mathcal{R}\text{en}(\mathcal{M})) \qquad \langle\varepsilon\rangle = \varepsilon^{\mathcal{M}}$$

$$\frac{\Gamma \in \mathrm{Obj}(\mathcal{R}\text{en}(\mathcal{M})) \qquad A \in \text{ty}^{\mathcal{M}}\,\langle\Gamma\rangle}{\Gamma.A \in \mathrm{Obj}(\mathcal{R}\text{en}(\mathcal{M}))} \qquad \langle\Gamma.A\rangle = \langle\Gamma\rangle .^{\mathcal{M}} A$$

$$\frac{\Gamma \in \mathrm{Obj}(\mathcal{R}\text{en}(\mathcal{M})) \qquad b \in \text{tm}^{\mathcal{M}}\,\langle\Gamma\rangle\ \text{Bool}^{\mathcal{M}} \qquad x \in \mathbb{B}}{\Gamma, b \stackrel{\text{def}}{=} x \in \mathrm{Obj}(\mathcal{R}\text{en}(\mathcal{M}))} \qquad \langle\Gamma, b \stackrel{\text{def}}{=} x\rangle = \langle\Gamma\rangle , b \stackrel{\text{def}}{=}^{\mathcal{M}} x$$

A yoga of 1st and 2nd order models

Combining constructions on models of (**SO**)**GAT**

- **Internalization** of a f.o model $\mathcal{M}$ as a s.o model $\mathbb{Int}(\mathcal{M})$ in Psh($\mathcal{M}$)
- **Contextualisation** of a s.o. model $\mathbb{M}$ to a first order model $\mathcal{T}$ele($\mathbb{M}$)
- **Pullback** along a functor $F : \mathcal{C} \to \mathcal{D}$ of a f.o. model $\mathcal{M}$ internal to Psh($\mathcal{D}$) into $F^*\mathcal{M}$ internal to Psh($\mathcal{C}$)
- **Sconing** of a s.o. model $\mathbb{M}^*$ displayed over a f.o. model $\mathcal{M}$ into a f.o model $\pi : \mathcal{S}\text{cone}(\mathbb{M}^*) \to \mathcal{M}$

A yoga of 1st and 2nd order models

Combining constructions on models of (**SO**)**GAT**

- **Internalization** of a f.o model $\mathcal{M}$ as a s.o model $\mathbb{Int}(\mathcal{M})$ in $\mathsf{Psh}(\mathcal{M})$
- **Contextualisation** of a s.o. model $\mathbb{M}$ to a first order model $\mathcal{T}\mathsf{ele}(\mathbb{M})$
- **Pullback** along a functor $F : \mathcal{C} \to \mathcal{D}$ of a f.o. model $\mathcal{M}$ internal to $\mathsf{Psh}(\mathcal{D})$ into $F^*\mathcal{M}$ internal to $\mathsf{Psh}(\mathcal{C})$
- **Sconing** of a s.o. model $\mathbb{M}^*$ displayed over a f.o. model $\mathcal{M}$ into a f.o model $\pi : \mathcal{S}\mathsf{cone}(\mathbb{M}^*) \to \mathcal{M}$

Relative induction with respect to renamings:

- Building sections of $\pi : \mathcal{S}\mathsf{cone}(\mathbb{M}^*) \to F^*\mathcal{T}\mathsf{ele}(\mathbb{Int}(\mathbb{0}_{\Pi\mathbb{B}^{\mathsf{ext}}_{\mathbf{GAT}}}))$ where $F : \mathcal{R}\mathsf{en}(\mathbb{0}_{\Pi\mathbb{B}^{\mathsf{ext}}_{\mathbf{GAT}}}) \to \mathbb{0}_{\Pi\mathbb{B}^{\mathsf{ext}}_{\mathbf{GAT}}}$