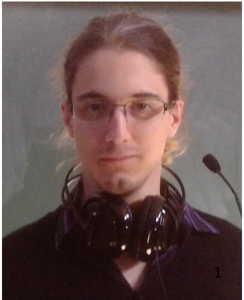
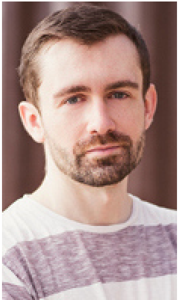
An introduction to polarised L calculi

Kenji Maillard, Étienne Miquey, Xavier Montillet,
Guillaume Munch-Maccagnoni, Gabriel Scherer

INRIA (Paris, Nantes, Saclay)

Séminaire Gallium, October 15th 2018


Why?

L: polarized versions of the $\lambda\mu\tilde{\mu}$ abstract-machine calculi.

Claim: a good **syntax** to talk about

- classical computation
- mixed-polarity equivalences
- effects
- purity

(Semantics: roughly in line with CBPV models.)

Related works:

- Zeilberger (direct relation to focusing; no syntax)
- CBPV (related models, worse syntax)

Section 1

$\mu\tilde{\mu}$ recap

Abstract machines

$$
\begin{array}{rcl}
\langle t\ u \parallel e \rangle & \triangleright_{\mathsf{abs}} & \langle t \parallel u \cdot e \rangle \\
\langle \lambda x.\, t \parallel u \cdot e \rangle & \triangleright_{\mathsf{abs}} & \langle t\,[u/x] \parallel e \rangle
\end{array}
$$

Abstract machines

$$\begin{array}{rcl} \langle t\ u \parallel e \rangle & \triangleright_{\text{abs}} & \langle t \parallel u \cdot e \rangle \\ \langle \lambda x.\, t \parallel u \cdot e \rangle & \triangleright_{\text{abs}} & \langle t\,[u/x] \parallel e \rangle \end{array}$$

$$c ::= \langle t \parallel e \rangle \quad \text{configuration}$$

$$\begin{array}{rlll} t, u ::= & & & \text{term} \\ & \mid & x, y, z & \text{variable} \\ & \mid & t\ u & \text{application} \\ & \mid & \lambda x.\, t & \lambda\text{-abstraction} \end{array}$$

$$\begin{array}{rlll} e, f ::= & & & \text{context} \\ & \mid & \star & \text{empty} \\ & \mid & t \cdot e & \text{application stack} \end{array}$$

Introducing μ

$$\langle t \; u \,\|\, e \rangle \rhd_{\text{abs}} \langle t \,\|\, u \cdot e \rangle$$

This reduction **defines** $(t \; u)$:

$$\langle t \; u \,| \quad : \quad |\, e \rangle \longmapsto \langle t \,\|\, u \cdot e \rangle$$

It is the term that, when put against $|\, e \rangle$*, reduces to* $\langle t \,\|\, u \cdot e \rangle$*.*

Introducing μ

$$\langle t\ u \,\|\, e\rangle \triangleright_{\mathsf{abs}} \langle t \,\|\, u \cdot e\rangle$$

This reduction **defines** $(t\ u)$:

$$\langle t\ u\,| \quad : \quad |\, e\rangle \longmapsto \langle t \,\|\, u \cdot e\rangle$$

It is the term that, when put against $|\, e\rangle$, *reduces to* $\langle t \,\|\, u \cdot e\rangle$.

Idea : introduces a **universal** solution to this problem

$$\mu\alpha. \langle t \,\|\, u \cdot \alpha\rangle$$

Introducing μ

$$\langle t\ u \,\|\, e\rangle \rhd_{\text{abs}} \langle t \,\|\, u \cdot e\rangle$$

This reduction **defines** $(t\ u)$:

$$\langle t\ u\,| \quad : \quad |\, e\rangle \longmapsto \langle t \,\|\, u \cdot e\rangle$$

It is the term that, when put against $|\, e\rangle$, *reduces to* $\langle t \,\|\, u \cdot e\rangle$.

Idea : introduces a **universal** solution to this problem

$$\mu\alpha.\,\langle t \,\|\, u \cdot \alpha\rangle$$

$$\langle \mu\alpha.\, c \,\|\, e\rangle \rhd_{\mu} c\,[e/\alpha]$$

Introducing μ

$$\langle t\ u \,\|\, e \rangle \rhd_{\text{abs}} \langle t \,\|\, u \cdot e \rangle$$

This reduction **defines** $(t\ u)$:

$$\langle t\ u \,| \quad : \quad |\, e \rangle \longmapsto \langle t \,\|\, u \cdot e \rangle$$

It is the term that, when put against $|\, e \rangle$*, reduces to* $\langle t \,\|\, u \cdot e \rangle$*.*

Idea : introduces a **universal** solution to this problem

$$\mu\alpha.\, \langle t \,\|\, u \cdot \alpha \rangle$$

$$\langle \mu\alpha.\, c \,\|\, e \rangle \rhd_{\mu} c\, [e/\alpha]$$

$$t\ u \quad \stackrel{\text{def}}{:=} \quad \mu\alpha.\, \langle t \,\|\, u \cdot \alpha \rangle$$

Detroducing λ

$$\langle \lambda x.\, t \parallel u \cdot e \rangle \triangleright_{\text{abs}} \langle t\,[u/x] \parallel e \rangle$$

Idea : λ pattern-matches on the context, deconstructing $u \cdot e$.

Detroducing λ

$$\langle \lambda x.\, t \parallel u \cdot e \rangle \triangleright_{\text{abs}} \langle t\,[u/x] \parallel e \rangle$$

Idea : λ pattern-matches on the context, deconstructing $u \cdot e$.

$$\mu(x \cdot \alpha).\, c$$

Detroducing λ

$$\langle \lambda x.\, t \parallel u \cdot e \rangle \triangleright_{\text{abs}} \langle t\,[u/x] \parallel e \rangle$$

Idea : λ pattern-matches on the context, deconstructing $u \cdot e$.

$$\mu(x \cdot \alpha).\, c$$

$$\langle \mu(x \cdot \alpha).\, c \parallel u \cdot e \rangle \triangleright_{\text{fun}} c\,[u/x, e/\alpha]$$

Detroducing λ

$$\langle \lambda x.\, t \parallel u \cdot e \rangle \triangleright_{\text{abs}} \langle t\,[u/x] \parallel e \rangle$$

Idea : λ pattern-matches on the context, deconstructing $u \cdot e$.

$$\mu(x \cdot \alpha).\, c$$

$$\langle \mu(x \cdot \alpha).\, c \parallel u \cdot e \rangle \triangleright_{\text{fun}} c\,[u/x, e/\alpha]$$

$$\lambda x.\, t \overset{\text{def}}{:=} \mu(x \cdot \alpha).\, \langle t \parallel \alpha \rangle$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$$\begin{aligned} t, u ::= \; & \\ & \mid x, y, z \\ & \mid \mu\alpha.\, c \\ & \mid \mu(x \cdot \alpha).\, c \\ & \mid (t, u) \end{aligned}$$

$$\begin{aligned} e, f ::= \; & \\ & \mid \star, \alpha, \beta \\ & \mid \\ & \mid t \cdot e \end{aligned}$$

$$\langle \text{let } (x_1, x_2) = t \text{ in } u \parallel e \rangle \triangleright_{\text{abs}} \big\langle t \,\big\|\, \text{let } (x_1, x_2) = \square \text{ in } \langle u \parallel e \rangle \big\rangle$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$t, u ::=$

$\mid x, y, z$

$\mid \mu\alpha.\, c$

$\mid \mu(x \cdot \alpha).\, c$

$\mid (t, u)$

$e, f ::=$

$\mid \star, \alpha, \beta$

$\mid$

$\mid t \cdot e$

$$\langle \text{let } (x_1, x_2) = t \text{ in } u \parallel e \rangle \triangleright_{\text{abs}} \left\langle t \,\middle\|\, \text{let } (x_1, x_2) = \square \text{ in } \langle u \parallel e \rangle \right\rangle$$

$$\text{let } (x_1, x_2) = t \text{ in } u \quad \stackrel{\text{def}}{:=} \quad \mu\alpha.\, \left\langle t \,\middle\|\, \tilde{\mu}(x_1, x_2).\, \langle u \parallel \alpha \rangle \right\rangle$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$t, u ::=$
$\mid x, y, z$
$\mid \mu\alpha.\, c$
$\mid \mu(x \cdot \alpha).\, c$
$\mid (t, u)$

$e, f ::=$
$\mid \star, \alpha, \beta$
$\mid$
$\mid t \cdot e$
$\mid \tilde{\mu}(x_1, x_2).\, c$

$$\langle \text{let } (x_1, x_2) = t \text{ in } u \parallel e \rangle \triangleright_{\text{abs}} \left\langle t \,\middle\|\, \text{let } (x_1, x_2) = \square \text{ in } \langle u \parallel e \rangle \right\rangle$$

$$\text{let } (x_1, x_2) = t \text{ in } u \quad \overset{\text{def}}{:=} \quad \mu\alpha.\, \left\langle t \,\middle\|\, \tilde{\mu}(x_1, x_2).\, \langle u \parallel \alpha \rangle \right\rangle$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$$
\begin{aligned}
t, u ::= & \\
& \mid x, y, z \\
& \mid \mu\alpha.\, c \\
& \mid \mu(x \cdot \alpha).\, c \\
& \mid (t, u)
\end{aligned}
\qquad
\begin{aligned}
e, f ::= & \\
& \mid \star, \alpha, \beta \\
& \mid ? \\
& \mid t \cdot e \\
& \mid \tilde{\mu}(x_1, x_2).\, c
\end{aligned}
$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$$\begin{aligned} t, u ::= & \\ & \mid x, y, z \\ & \mid \mu\alpha.\, c \\ & \mid \mu(x \cdot \alpha).\, c \\ & \mid (t, u) \end{aligned}$$

$$\begin{aligned} e, f ::= & \\ & \mid \star, \alpha, \beta \\ & \mid \tilde{\mu}x.\, c \\ & \mid t \cdot e \\ & \mid \tilde{\mu}(x_1, x_2).\, c \end{aligned}$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$$\begin{aligned} t, u ::= & \\ & \mid x, y, z \\ & \mid \mu\alpha.\, c \\ & \mid \mu(x \cdot \alpha).\, c \\ & \mid (t, u) \end{aligned} \qquad \begin{aligned} e, f ::= & \\ & \mid \star, \alpha, \beta \\ & \mid \tilde{\mu}x.\, c \\ & \mid t \cdot e \\ & \mid \tilde{\mu}(x_1, x_2).\, c \end{aligned}$$

$$\begin{aligned} \langle \mu\alpha.\, c \parallel e \rangle & \quad \triangleright_{\mu} \quad c\,[e/\alpha] \\ \langle t \parallel \tilde{\mu}x.\, c \rangle & \quad \triangleright_{\tilde{\mu}} \quad c\,[t/x] \\ \langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2).\, c \rangle & \quad \triangleright_{\otimes} \quad c\,[t_1/x_1, t_2/x_2] \\ \langle \mu(x \cdot \alpha).\, c \parallel t \cdot e \rangle & \quad \triangleright_{\rightarrow} \quad c\,[t/x, e/\alpha] \end{aligned}$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$$\begin{aligned} t, u ::= &\\ &\mid x, y, z\\ &\mid \mu\alpha.\, c\\ &\mid \mu(x \cdot \alpha).\, c\\ &\mid (t, u)\\ &\mid \sigma_i\ t \end{aligned}$$

$$\begin{aligned} e, f ::= &\\ &\mid \star, \alpha, \beta\\ &\mid \tilde{\mu}x.\, c\\ &\mid t \cdot e\\ &\mid \tilde{\mu}(x_1, x_2).\, c \end{aligned}$$

$$\begin{aligned} \langle \mu\alpha.\, c \parallel e \rangle &\quad \rhd_{\mu} \quad c\,[e/\alpha]\\ \langle t \parallel \tilde{\mu}x.\, c \rangle &\quad \rhd_{\tilde{\mu}} \quad c\,[t/x]\\ \langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2).\, c \rangle &\quad \rhd_{\otimes} \quad c\,[t_1/x_1, t_2/x_2]\\ \langle \mu(x \cdot \alpha).\, c \parallel t \cdot e \rangle &\quad \rhd_{\rightarrow} \quad c\,[t/x, e/\alpha] \end{aligned}$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$$
\begin{aligned}
t, u ::= \;& \\
& \mid x, y, z \\
& \mid \mu\alpha.\, c \\
& \mid \mu(x \cdot \alpha).\, c \\
& \mid (t, u) \\
& \mid \sigma_i\ t
\end{aligned}
\qquad
\begin{aligned}
e, f ::= \;& \\
& \mid \star, \alpha, \beta \\
& \mid \tilde{\mu}x.\, c \\
& \mid t \cdot e \\
& \mid \tilde{\mu}(x_1, x_2).\, c \\
& \mid \tilde{\mu}[(\sigma_1\ x_1).\, c_1 \mid (\sigma_2\ x_2).\, c_2]
\end{aligned}
$$

$$
\begin{aligned}
\langle \mu\alpha.\, c \parallel e \rangle &\quad \rhd_{\mu} \quad c\,[e/\alpha] \\
\langle t \parallel \tilde{\mu}x.\, c \rangle &\quad \rhd_{\tilde{\mu}} \quad c\,[t/x] \\
\langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2).\, c \rangle &\quad \rhd_{\otimes} \quad c\,[t_1/x_1, t_2/x_2] \\
\langle \mu(x \cdot \alpha).\, c \parallel t \cdot e \rangle &\quad \rhd_{\to} \quad c\,[t/x, e/\alpha]
\end{aligned}
$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$$t, u ::= \;|\; x, y, z \;|\; \mu\alpha.\, c \;|\; \mu(x \cdot \alpha).\, c \;|\; (t, u) \;|\; \sigma_i\ t$$

$$e, f ::= \;|\; \star, \alpha, \beta \;|\; \tilde{\mu}x.\, c \;|\; t \cdot e \;|\; \tilde{\mu}(x_1, x_2).\, c \;|\; \tilde{\mu}[(\sigma_1\ x_1).\, c_1 \mid (\sigma_2\ x_2).\, c_2] \;|\; \pi_i\ e$$

$$\begin{array}{rcl}
\langle \mu\alpha.\, c \parallel e \rangle & \rhd_{\mu} & c\,[e/\alpha] \\
\langle t \parallel \tilde{\mu}x.\, c \rangle & \rhd_{\tilde{\mu}} & c\,[t/x] \\
\langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2).\, c \rangle & \rhd_{\otimes} & c\,[t_1/x_1, t_2/x_2] \\
\langle \mu(x \cdot \alpha).\, c \parallel t \cdot e \rangle & \rhd_{\to} & c\,[t/x, e/\alpha]
\end{array}$$

Other datatypes and $\tilde{\mu}$

$$c ::= \langle t \parallel e \rangle \quad \text{command}$$

$$
\begin{aligned}
t, u ::= \;& \\
& \mid x, y, z \\
& \mid \mu\alpha.\, c \\
& \mid \mu(x \cdot \alpha).\, c \\
& \mid (t, u) \\
& \mid \sigma_i\ t \\
& \mid \mu[(\pi_1\ x_1).\, c_1 \mid (\pi_2\ x_2).\, c_2]
\end{aligned}
$$

$$
\begin{aligned}
e, f ::= \;& \\
& \mid \star, \alpha, \beta \\
& \mid \tilde{\mu}x.\, c \\
& \mid t \cdot e \\
& \mid \tilde{\mu}(x_1, x_2).\, c \\
& \mid \tilde{\mu}[(\sigma_1\ x_1).\, c_1 \mid (\sigma_2\ x_2).\, c_2] \\
& \mid \pi_i\ e
\end{aligned}
$$

$$
\begin{aligned}
\langle \mu\alpha.\, c \parallel e \rangle &\quad \rhd_{\mu} \quad c\,[e/\alpha] \\
\langle t \parallel \tilde{\mu}x.\, c \rangle &\quad \rhd_{\tilde{\mu}} \quad c\,[t/x] \\
\langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2).\, c \rangle &\quad \rhd_{\otimes} \quad c\,[t_1/x_1, t_2/x_2] \\
\langle \mu(x \cdot \alpha).\, c \parallel t \cdot e \rangle &\quad \rhd_{\to} \quad c\,[t/x, e/\alpha]
\end{aligned}
$$

Program equivalences

The η-expansion rules are beautiful and regular.
(Thank you, sequent calculus.)

In the λ-calculus:

$$\begin{array}{rrcl} & (t : A \to B) & \rhd_\eta & \lambda x.\, t\ x \\ \forall C, & C[t : A \otimes B] & \rhd_\eta & \text{let } (x_1, x_2) = t \text{ in } C[(x_1, x_2)] \end{array}$$

In L:

$$\begin{array}{rcl} (t : A \to B) & \rhd_\eta & \mu(x \cdot \alpha).\, \langle t \,\|\, x \cdot \alpha \rangle \\ (e : A \otimes B) & \rhd_\eta & \tilde{\mu}(x_1, x_2).\, \langle (x_1, x_2) \,\|\, e \rangle \end{array}$$

Program equivalences

The η-expansion rules are beautiful and regular.
(Thank you, sequent calculus.)

In the λ-calculus:

$$\begin{array}{rrcl} & (t : A \to B) & \rhd_\eta & \lambda x.\, t\ x \\ \forall C, & C[t : A \otimes B] & \rhd_\eta & \text{let } (x_1, x_2) = t \text{ in } C[(x_1, x_2)] \end{array}$$

In L:

$$\begin{array}{rcl} (t : A \to B) & \rhd_\eta & \mu(x \cdot \alpha).\, \langle t \parallel x \cdot \alpha \rangle \\ (e : A \otimes B) & \rhd_\eta & \tilde{\mu}(x_1, x_2).\, \langle (x_1, x_2) \parallel e \rangle \end{array}$$

Killer feature!

Type system...

$$\begin{array}{rclllll} t, u & ::= & x, y, z & \mid \mu\alpha.\, c & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \star, \alpha, \beta & \mid \tilde{\mu}x.\, c & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

$$c ::= \langle t \parallel e \rangle$$

Type system...

$$\begin{array}{llllll} t, u & ::= & x, y, z & \mid \mu\alpha.\, c & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \star, \alpha, \beta & \mid \tilde{\mu}x.\, c & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

$$c ::= \langle t \parallel e \rangle \qquad \boxed{c : (\Gamma \vdash \Delta)} \qquad \boxed{\Gamma \vdash t : A \mid \Delta} \qquad \boxed{\Gamma \mid e : A \vdash \Delta}$$

Type system...

$$
\begin{array}{lcllll}
t, u & ::= & x, y, z & \mid \mu\alpha.\, c & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\
e, f & ::= & \star, \alpha, \beta & \mid \tilde{\mu}x.\, c & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

$$
c ::= \langle t \parallel e \rangle \qquad \boxed{c : (\Gamma \vdash \Delta)} \qquad \boxed{\Gamma \vdash t : A \mid \Delta} \qquad \boxed{\Gamma \mid e : A \vdash \Delta}
$$

$$
\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma \mid e : A \vdash \Delta}{\langle t \parallel e \rangle : (\Gamma \vdash \Delta)} \qquad \frac{}{\Gamma, x : A \vdash x : A \mid \Delta} \qquad \frac{}{\Gamma \mid \alpha : A \vdash \alpha : A, \Delta}
$$

Type system...

$$
\begin{array}{llllll}
t, u & ::= & x, y, z & \mid \mu\alpha.\, c & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\
e, f & ::= & \star, \alpha, \beta & \mid \tilde{\mu}x.\, c & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

$$
c ::= \langle t \parallel e \rangle \qquad \boxed{c : (\Gamma \vdash \Delta)} \qquad \boxed{\Gamma \vdash t : A \mid \Delta} \qquad \boxed{\Gamma \mid e : A \vdash \Delta}
$$

$$
\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma \mid e : A \vdash \Delta}{\langle t \parallel e \rangle : (\Gamma \vdash \Delta)} \qquad \frac{}{\Gamma, x : A \vdash x : A \mid \Delta} \qquad \frac{}{\Gamma \mid \alpha : A \vdash \alpha : A, \Delta}
$$

$$
\frac{c : (\Gamma, x_1 : A_1, x_2 : A_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x_1, x_2).\, c : A_1 \otimes A_2 \vdash \Delta} \qquad \frac{\Gamma \vdash t_1 : A_1 \mid \Delta \qquad \Gamma \vdash t_2 : A_2 \mid \Delta}{\Gamma \vdash (t_1, t_2) : A_1 \otimes A_2 \mid \Delta}
$$

Type system...

$$
\begin{array}{rcllll}
t, u & ::= & x, y, z & \mid \mu\alpha.\, c & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\
e, f & ::= & \star, \alpha, \beta & \mid \tilde{\mu}x.\, c & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

$$
c ::= \langle t \parallel e \rangle \qquad \boxed{c : (\Gamma \vdash \Delta)} \qquad \boxed{\Gamma \vdash t : A \mid \Delta} \qquad \boxed{\Gamma \mid e : A \vdash \Delta}
$$

$$
\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma \mid e : A \vdash \Delta}{\langle t \parallel e \rangle : (\Gamma \vdash \Delta)} \qquad \frac{}{\Gamma, x : A \vdash x : A \mid \Delta} \qquad \frac{}{\Gamma \mid \alpha : A \vdash \alpha : A, \Delta}
$$

$$
\frac{c : (\Gamma, x_1 : A_1, x_2 : A_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x_1, x_2).\, c : A_1 \otimes A_2 \vdash \Delta} \qquad \frac{\Gamma \vdash t_1 : A_1 \mid \Delta \qquad \Gamma \vdash t_2 : A_2 \mid \Delta}{\Gamma \vdash (t_1, t_2) : A_1 \otimes A_2 \mid \Delta}
$$

$$
\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma \mid e : B \vdash \Delta}{\Gamma \mid t \cdot e : A \to B \vdash \Delta} \qquad \frac{c : (\Gamma, x : A \vdash \alpha : B, \Delta)}{\Gamma \vdash \mu(x \cdot \alpha).\, c : A \to B \mid \Delta}
$$

Type system...

$$
\begin{array}{lclllll}
t, u & ::= & x, y, z & \mid \mu\alpha.\, c & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\
e, f & ::= & \star, \alpha, \beta & \mid \tilde{\mu}x.\, c & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

$$c ::= \langle t \parallel e \rangle \qquad \boxed{c : (\Gamma \vdash \Delta)} \qquad \boxed{\Gamma \vdash t : A \mid \Delta} \qquad \boxed{\Gamma \mid e : A \vdash \Delta}$$

$$\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma \mid e : A \vdash \Delta}{\langle t \parallel e \rangle : (\Gamma \vdash \Delta)} \qquad \frac{}{\Gamma, x : A \vdash x : A \mid \Delta} \qquad \frac{}{\Gamma \mid \alpha : A \vdash \alpha : A, \Delta}$$

$$\frac{c : (\Gamma, x_1 : A_1, x_2 : A_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x_1, x_2).\, c : A_1 \otimes A_2 \vdash \Delta} \qquad \frac{\Gamma \vdash t_1 : A_1 \mid \Delta \qquad \Gamma \vdash t_2 : A_2 \mid \Delta}{\Gamma \vdash (t_1, t_2) : A_1 \otimes A_2 \mid \Delta}$$

$$\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma \mid e : B \vdash \Delta}{\Gamma \mid t \cdot e : A \to B \vdash \Delta} \qquad \frac{c : (\Gamma, x : A \vdash \alpha : B, \Delta)}{\Gamma \vdash \mu(x \cdot \alpha).\, c : A \to B \mid \Delta}$$

$$\frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha.\, c : A \mid \Delta} \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \mid \tilde{\mu}x.\, c : A \vdash \Delta}$$

and logic

$$\boxed{\Gamma \vdash t : A \mid \Delta} \qquad \boxed{\Gamma \mid e : A \vdash \Delta} \qquad \boxed{c : (\Gamma \vdash \Delta)}$$

Multiple co-variables $\longleftrightarrow$ **classical** logic

$$\text{call/cc}(f) \quad \overset{\text{def}}{:=}$$

and logic

$$\boxed{\Gamma \vdash t : A \mid \Delta} \qquad \boxed{\Gamma \mid e : A \vdash \Delta} \qquad \boxed{c : (\Gamma \vdash \Delta)}$$

Multiple co-variables $\longleftrightarrow$ **classical** logic

$$\mathrm{call/cc}(f) \quad \stackrel{\text{def}}{:=} \quad \mu\alpha.\left\langle f \,\middle\|\, (\mu(x \cdot \beta).\langle x \,\|\, \alpha\rangle) \cdot \alpha \right\rangle$$

and logic

$$\boxed{\Gamma \vdash t : A \mid \Delta} \qquad \boxed{\Gamma \mid e : A \vdash \Delta} \qquad \boxed{c : (\Gamma \vdash \Delta)}$$

Multiple co-variables $\longleftrightarrow$ **classical** logic

$$\mathrm{call/cc}(f) \quad \overset{\text{def}}{:=} \quad \mu\alpha.\, \langle f \,\|\, (\mu(x \cdot \beta).\, \langle x \,\|\, \alpha\rangle) \cdot \alpha \rangle$$

Intuitionistic restrictions, either:

- linear co-context

$$\frac{}{\Gamma, x : A \vdash x : A \mid \emptyset} \qquad \frac{\Gamma \vdash t : A \mid \Delta_1 \qquad \Gamma \mid e : B \vdash \Delta_2}{\Gamma \mid t \cdot e : A \to B \vdash \Delta_1, \Delta_2}$$

- single variable + shadowing

$$t\ u \overset{\text{def}}{:=} \mu\alpha.\, \langle t \,\|\, u \cdot \alpha \rangle \qquad \lambda x.\, t \overset{\text{def}}{:=} \mu(x \cdot \alpha).\, \langle t \,\|\, \alpha \rangle$$

$$t\ u \overset{\text{def}}{:=} \mu \star .\, \langle t \,\|\, u \cdot \star \rangle \qquad \lambda x.\, t \overset{\text{def}}{:=} \mu(x \cdot \star).\, \langle t \,\|\, \star \rangle$$

Confluence problem

Computational Classical logic: hard !

$$\langle \mu\alpha.\, c_1 \,\|\, \tilde{\mu}x.\, c_2 \rangle$$

Confluence problem

Computational Classical logic: hard !

$$\langle \mu\alpha.\, c_1 \,\|\, \tilde{\mu}x.\, c_2 \rangle \quad \triangleright \quad c_2\,[\mu\alpha.\, c_1 / x]$$

Confluence problem

Computational Classical logic: hard !

$$c_1\,[\tilde{\mu}x.\,c_2/\alpha] \quad \triangleleft \quad \langle \mu\alpha.\,c_1 \,\|\, \tilde{\mu}x.\,c_2 \rangle \quad \triangleright \quad c_2\,[\mu\alpha.\,c_1/x]$$

Confluence problem

Computational Classical logic: hard !

$$c_1\,[\tilde{\mu}x.\,c_2/\alpha] \quad \triangleleft \quad \langle \mu\alpha.\,c_1 \,\|\, \tilde{\mu}x.\,c_2 \rangle \quad \triangleright \quad c_2\,[\mu\alpha.\,c_1/x]$$

$$\langle x \,\|\, \alpha \rangle \quad \triangleleft \quad \big\langle \mu_{-}.\,\langle x \,\|\, \alpha \rangle \,\big\|\, \tilde{\mu}_{-}.\,\langle y \,\|\, \beta \rangle \big\rangle \quad \triangleright \quad \langle y \,\|\, \beta \rangle$$

Confluence problem

Computational Classical logic: hard !

$$c_1\,[\tilde{\mu}x.\,c_2/\alpha] \quad \triangleleft \quad \langle \mu\alpha.\,c_1 \,\|\, \tilde{\mu}x.\,c_2 \rangle \quad \triangleright \quad c_2\,[\mu\alpha.\,c_1/x]$$

$$\langle x \,\|\, \alpha \rangle \quad \triangleleft \quad \Big\langle \mu\_.\,\langle x \,\|\, \alpha \rangle \,\Big\|\, \tilde{\mu}\_.\,\langle y \,\|\, \beta \rangle \Big\rangle \quad \triangleright \quad \langle y \,\|\, \beta \rangle$$

Arbitrary resolution of the critical pair:

- prefer reducing the term: call-by-value
- prefer reducing the co-term: call-by-name

Confluence problem: values and co-values

$$\langle \mu\alpha.\, c_1 \parallel \tilde{\mu}x.\, c_2 \rangle$$

$$\begin{array}{lcllll} t, u & ::= & \mu\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Confluence problem: values and co-values

$$\langle \mu\alpha.\, c_1 \,\|\, \tilde{\mu}x.\, c_2 \rangle$$

$$\begin{array}{lllll} t, u & ::= & \mu\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Prefer reducing the term: $\langle V \,\|\, \tilde{\mu}x.\, c \rangle \rhd_{\tilde{\mu}} c\,[V/x]$

Confluence problem: values and co-values

$$\langle \mu\alpha.\, c_1 \,\|\, \tilde{\mu}x.\, c_2 \rangle$$

$$\begin{array}{rcllll} t, u & ::= & \mu\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Prefer reducing the term: $\langle V \,\|\, \tilde{\mu}x.\, c \rangle \rhd_{\tilde{\mu}} c\,[V/x]$

$$\begin{array}{rcllll} t, u & ::= & \mu\alpha.\, c & \mid V & & \\ V, W & ::= & & x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Confluence problem: values and co-values

$$\langle \mu\alpha.\, c_1 \parallel \tilde{\mu}x.\, c_2 \rangle$$

$$\begin{array}{rcllll} t, u & ::= & \mu\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Prefer reducing the term: $\langle V \parallel \tilde{\mu}x.\, c \rangle \rhd_{\tilde{\mu}} c\,[V/x]$

$$\begin{array}{rcllll} t, u & ::= & \mu\alpha.\, c & \mid V & & \\ V, W & ::= & & x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Prefer reducing the co-term: $\langle \mu\alpha.\, c \parallel S \rangle \rhd_{\mu} c\,[S/\alpha]$

Confluence problem: values and co-values

$$\langle \mu\alpha.\, c_1 \parallel \tilde{\mu}x.\, c_2 \rangle$$

$$\begin{array}{lcllll} t, u & ::= & \mu\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Prefer reducing the term: $\langle V \parallel \tilde{\mu}x.\, c \rangle \rhd_{\tilde{\mu}} c\,[V/x]$

$$\begin{array}{lcllll} t, u & ::= & \mu\alpha.\, c & \mid V & & \\ V, W & ::= & & x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Prefer reducing the co-term: $\langle \mu\alpha.\, c \parallel S \rangle \rhd_{\mu} c\,[S/\alpha]$

$$\begin{array}{lcllll} t, u & ::= & \mu\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid S & & \\ S & ::= & & \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Pointers

The duality of computation (Curien and Herbelin, 2000)
introduced the syntax (with λ)

Tutorial on computational classical logic and the sequent calculus
(Downen and Ariola, 2018) (in JFP)

Examples of PL research it inspired:

- Copatterns: Programming Infinite Structures by Observations
  Abel, Pientka, Thibodeau, and Setzer (2013)
- Structures for Structural Recursion
  Downen, Johnson-Freyd, and Ariola (2015)
- A classical sequent calculus with dependent types
  Miquey (2017)
- Deciding equivalence with sums and the empty type
  Scherer (2017)

Section 2

A bit of polarity

Call-by-value,name restrictions: **global** confluence solutions.

$$
\begin{array}{rcllll}
t, u & ::= & \mu\alpha.\, c & \mid V & & \\
V, W & ::= & & x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\
e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

Call-by-value,name restrictions: **global** confluence solutions.

$$
\begin{array}{rcllll}
t, u & ::= & \mu\alpha.\, c & \mid V & & \\
V, W & ::= & & x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\
e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

Polarity: a **type-directed** solution.

$$
\begin{array}{rcllllll}
P, Q & ::= & X^+, Y^+ & \mid P \otimes Q & \mid P \oplus Q & \mid \dots & \mid \langle N \rangle^+ \\
M, N & ::= & X^-, Y^- & \mid P \to N & \mid M \times N & \mid \dots & \mid \langle P \rangle^-
\end{array}
$$

Call-by-value,name restrictions: **global** confluence solutions.

$$
\begin{array}{rcllll}
t, u & ::= & \mu\alpha.\, c & \mid V & & \\
V, W & ::= & & x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\
e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

Polarity: a **type-directed** solution.

$$
\begin{array}{rcllllll}
P, Q & ::= & X^+, Y^+ & \mid P \otimes Q & \mid P \oplus Q & \mid \ldots & \mid \langle N \rangle^+ \\
M, N & ::= & X^-, Y^- & \mid P \to N & \mid M \times N & \mid \ldots & \mid \langle P \rangle^-
\end{array}
$$

$$
c \quad ::= \quad \langle t \parallel e \rangle^- \quad \mid \langle t \parallel e \rangle^+
$$

Call-by-value,name restrictions: **global** confluence solutions.

$$\begin{array}{rcllll} t, u & ::= & \mu\alpha.\, c & \mid V & & \\ V, W & ::= & & x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\ e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c \end{array}$$

Polarity: a **type-directed** solution.

$$\begin{array}{rclllll} P, Q & ::= & X^+, Y^+ & \mid P \otimes Q & \mid P \oplus Q & \mid \dots & \mid \langle N \rangle^+ \\ M, N & ::= & X^-, Y^- & \mid P \to N & \mid M \times N & \mid \dots & \mid \langle P \rangle^- \end{array}$$

Negative terms (by-name) are inert values.

$$\begin{array}{rcllll} c & ::= & \langle V \parallel e \rangle^- & \mid \langle t \parallel e \rangle^+ & & \\ t, u & ::= & \mu^+\alpha.\, c & \mid V & & \\ V, W & ::= & \mu^-\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (V, W) \end{array}$$

Call-by-value,name restrictions: **global** confluence solutions.

$$\begin{array}{rcllll}
t, u & ::= & \mu\alpha.\, c & \mid V & & \\
V, W & ::= & & x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (t, u) \\
e, f & ::= & \tilde{\mu}x.\, c & \mid \star, \alpha, \beta & \mid t \cdot e & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}$$

Polarity: a **type-directed** solution.

$$\begin{array}{rclllll}
P, Q & ::= & X^+, Y^+ & \mid P \otimes Q & \mid P \oplus Q & \mid \ldots & \mid \langle N \rangle^+ \\
M, N & ::= & X^-, Y^- & \mid P \to N & \mid M \times N & \mid \ldots & \mid \langle P \rangle^-
\end{array}$$

Negative terms (by-name) are inert values.
Positive co-terms (by-value) are inert co-values.

$$\begin{array}{rcllll}
c & ::= & \langle V \,\|\, e \rangle^- & \mid \langle t \,\|\, S \rangle^+ & & \\
t, u & ::= & \mu^+\alpha.\, c & \mid V & & \\
V, W & ::= & \mu^-\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (V, W) \\
e, f & ::= & \tilde{\mu}^-x.\, c & \mid S & & \\
S & ::= & \tilde{\mu}^+x.\, c & \mid \star, \alpha, \beta & \mid V \cdot S & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}$$

Polarized reduction

$$
\begin{array}{rcl}
\langle \mu\alpha.\, c \parallel e \rangle & \rhd_{\mu} & c\,[e/\alpha] \\
\langle t \parallel \tilde{\mu}x.\, c \rangle & \rhd_{\tilde{\mu}} & c\,[t/x] \\
\langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2).\, c \rangle & \rhd_{\otimes} & c\,[t_1/x_1, t_2/x_2] \\
\langle \mu(x \cdot \alpha).\, c \parallel t \cdot e \rangle & \rhd_{\rightarrow} & c\,[t/x, e/\alpha]
\end{array}
$$

Polarized reduction

$$
\begin{array}{rcl}
\langle \mu\alpha.\, c \parallel e \rangle & \rhd_{\mu} & c\,[e/\alpha] \\
\langle t \parallel \tilde{\mu}x.\, c \rangle & \rhd_{\tilde{\mu}} & c\,[t/x] \\
\langle (t_1, t_2) \parallel \tilde{\mu}(x_1, x_2).\, c \rangle & \rhd_{\otimes} & c\,[t_1/x_1, t_2/x_2] \\
\langle \mu(x \cdot \alpha).\, c \parallel t \cdot e \rangle & \rhd_{\to} & c\,[t/x, e/\alpha]
\end{array}
$$

$$
\begin{array}{rcl}
\langle \mu\alpha.\, c \parallel S \rangle & \rhd_{\mu} & c\,[S/\alpha] \\
\langle V \parallel \tilde{\mu}x.\, c \rangle & \rhd_{\tilde{\mu}} & c\,[V/x] \\
\langle (V_1, V_2) \parallel \tilde{\mu}(x_1, x_2).\, c \rangle & \rhd_{\otimes} & c\,[V_1/x_1, V_2/x_2] \\
\langle \mu(x \cdot \alpha).\, c \parallel V \cdot S \rangle & \rhd_{\to} & c\,[V/x, S/\alpha]
\end{array}
$$

Strong values (Focalization)

$$
\begin{array}{rclllll}
c & ::= & \langle V \parallel e \rangle^- & \mid \langle t \parallel S \rangle^+ & & & \\
t, u & ::= & \mu^+\alpha.\, c & \mid V & & & \\
V, W & ::= & \mu^-\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (V, W) \\
e, f & ::= & \tilde{\mu}^- x.\, c & \mid S & & & \\
S & ::= & \tilde{\mu}^+ x.\, c & \mid \star, \alpha, \beta & \mid V \cdot S & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

Strong values (Focalization)

$$
\begin{array}{rcllll}
c & ::= & \langle V \parallel e \rangle^- & \mid \langle t \parallel S \rangle^+ & & \\
t, u & ::= & \mu^+ \alpha.\, c & \mid V & & \\
V, W & ::= & \mu^- \alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (V, W) \\
e, f & ::= & \tilde{\mu}^- x.\, c & \mid S & & \\
S & ::= & \tilde{\mu}^+ x.\, c & \mid \star, \alpha, \beta & \mid V \cdot S & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

What about (t, u) or $t \cdot e$ when t (for example) is not a value?

Strong values (Focalization)

$$
\begin{array}{rcllll}
c & ::= & \langle V \,\|\, e\rangle^{-} & \mid \langle t \,\|\, S\rangle^{+} & & \\
t, u & ::= & \mu^{+}\alpha.\, c & \mid V & & \\
V, W & ::= & \mu^{-}\alpha.\, c & \mid x, y, z & \mid \mu(x \cdot \alpha).\, c & \mid (V, W) \\
e, f & ::= & \tilde{\mu}^{-}x.\, c & \mid S & & \\
S & ::= & \tilde{\mu}^{+}x.\, c & \mid \star, \alpha, \beta & \mid V \cdot S & \mid \tilde{\mu}(x_1, x_2).\, c
\end{array}
$$

What about (t, u) or $t \cdot e$ when t (for example) is not a value?
Two design options:

- Force the user to decide an evaluation order.

$$
\big\langle t \,\big\|\, \tilde{\mu}x.\, \langle (x, u) \,\|\, \ldots \rangle \big\rangle \qquad\qquad \big\langle t \,\big\|\, \tilde{\mu}\langle \ldots \,\|\, x \cdot e \rangle.\, \big\rangle
$$

- Add an automatic reduction with arbitrary order: Wadler's ζ-rules.

$$
\langle (t, u) \,\|\, S\rangle^{+} \overset{t \notin V}{\triangleright_{\zeta.\mathrm{pair}.1}} \big\langle t \,\big\|\, \tilde{\mu}x.\, \langle (x, u) \,\|\, S\rangle^{+} \big\rangle
$$

Polarity: example

Recall : $t\ u \stackrel{\text{def}}{:=} \mu\alpha.\,\langle t \,\|\, u \cdot \alpha\rangle^{-}$

Consider $(\lambda x.\, t)\ u$:

$$\langle \lambda x.\, t \,\|\; u \;\cdot \star\rangle^{-}$$

Polarity: example

Recall : $t\ u \stackrel{\text{def}}{:=} \mu\alpha.\, \langle t \parallel u \cdot \alpha \rangle^-$

Consider $(\lambda x.\, t)\ u$:

$$\langle \lambda x.\, t \parallel u \cdot \star \rangle^-$$

Polarity: example

Recall : $t\ u \overset{\text{def}}{:=} \mu\alpha.\, \langle t \parallel u \cdot \alpha \rangle^{-}$

Consider $(\lambda x.\, t)\ u$:

$$\left\langle u \,\middle\|\, \tilde{\mu} y.\, \langle \lambda x.\, t \parallel y \cdot \star \rangle^{-} \right\rangle \qquad (\bullet)$$

Polarity: example

Recall : $t\ u \stackrel{\text{def}}{:=} \mu\alpha.\langle t \parallel u \cdot \alpha\rangle^{-}$

Consider $(\lambda x.\, t)\ u$:

$$\left\langle u \,\middle\|\, \tilde{\mu} y.\langle \lambda x.\, t \parallel y \cdot \star\rangle^{-} \right\rangle \qquad (\bullet)$$

u has a **negative** type

$\rightsquigarrow$ u is a value,

$$(\bullet) \triangleright^{*} t\,[u/x]$$

u has a **positive** type

$\rightsquigarrow$ u is not a value,

$$(\bullet) \triangleright^{*} \left\langle V \,\middle\|\, \tilde{\mu} y.\langle \lambda x.\, t \parallel y \cdot \star\rangle^{-} \right\rangle$$

$$\triangleright^{*} t\,[V/x]$$

Polarity: example

Recall : $t\ u \stackrel{\text{def}}{:=} \mu\alpha.\,\langle t \parallel u \cdot \alpha\rangle^-$

Consider $(\lambda x.\,t)\ u$:

$$\Big\langle u \,\Big\|\, \tilde{\mu}y.\,\langle \lambda x.\,t \parallel y \cdot \star\rangle^- \Big\rangle \qquad (\bullet)$$

u has a **negative** type
$\rightsquigarrow$ u is a value,

$$(\bullet) \rhd^* t\,[u/x]$$

u has a **positive** type
$\rightsquigarrow$ u is not a value,

$$(\bullet) \rhd^* \Big\langle V \,\Big\|\, \tilde{\mu}y.\,\langle \lambda x.\,t \parallel y \cdot \star\rangle^- \Big\rangle$$
$$\rhd^* t\,[V/x]$$

$P \to N$: call-by-value function
$\langle M\rangle^+ \to N$: call-by-name function

Polarity: summary

We name "L" the polarized $\mu\tilde{\mu}$ calcul$\{$us,i$\}$.

A **type-directed** resolution of classical non-confluence.
Lets you mix call-by-value and call-by-name (like CBPV).

Polarity: summary

We name "L" the polarized $\mu\tilde{\mu}$ calcul{us,i}.

A **type-directed** resolution of classical non-confluence.
Lets you mix call-by-value and call-by-name (like CBPV).

Let's make a bolder claim:

Polarity: summary

We name "L" the polarized $\mu\tilde{\mu}$ calcul{us,i}.

A **type-directed** resolution of classical non-confluence.
Lets you mix call-by-value and call-by-name (like CBPV).

Let's make a bolder claim:

The computational interpretation of classical logic.

Polarity: summary

We name "L" the polarized $\mu\tilde{\mu}$ calcul$\{$us,i$\}$.

A **type-directed** resolution of classical non-confluence.
Lets you mix call-by-value and call-by-name (like CBPV).

Let's make a bolder claim:

The computational interpretation of classical logic.

Constructive!

Polarity: summary

We name "L" the polarized $\mu\tilde{\mu}$ calcul{us,i}.

A **type-directed** resolution of classical non-confluence.
Lets you mix call-by-value and call-by-name (like CBPV).

Let's make a bolder claim:

The computational interpretation of classical logic.

Constructive!

Pointers : Girard's LC ('91), Munch-Maccagnoni's PhD thesis (2013)/(LICS'14).

Section 3

Effects & Purity

(Inspired by Paul's nice CBPV examples)

How to add a primitive effect to L?

Head reduction can be seen as a function

$$(\triangleright) : \mathsf{Com} \to \mathsf{Com}$$

written as a relation: $c_1 \triangleright c_2$ for $(\triangleright\, c_1) = c_2$.
(normal commands reduce into themselves)

How to add a primitive effect to L?

Head reduction can be seen as a function

$$(\triangleright) : \mathsf{Com} \to \mathsf{Com}$$

written as a relation: $c_1 \triangleright c_2$ for $(\triangleright\, c_1) = c_2$.
(normal commands reduce into themselves)

Now pick an effect T as a monad on Set, and extend

$$(\triangleright_T) : \mathsf{Com} \to T(\mathsf{Com})$$

Printing

Output literals $o \in \mathsf{O}$.

$$T(X) \stackrel{\text{def}}{:=} \mathsf{O}^* \times X$$

Printing

Output literals $o \in \mathsf{O}$.

$$T(X) \stackrel{\text{def}}{:=} \mathsf{O}^* \times X$$

$$c ::= \cdots \mid \mathsf{print}(o).\, c$$

Printing

Output literals $o \in \mathsf{O}$.

$$T(X) \stackrel{\text{def}}{:=} \mathsf{O}^* \times X$$

$$c ::= \cdots \mid \mathsf{print}(o).\, c$$

$$\mathsf{print}(o).\, c \rhd_T ([o], c)$$

Printing

Output literals $o \in \mathsf{O}$.

$$T(X) \overset{\text{def}}{:=} \mathsf{O}^* \times X$$

$$c ::= \cdots \mid \mathsf{print}(o).\, c$$

$$\mathsf{print}(o).\, c \rhd_T ([o], c)$$

Repeated reductions, Kleisli composition :

$$\mathsf{print}(1).\, \mathsf{print}(2).\, c \quad \rhd_T \quad ([1], \mathsf{print}(2).\, c) \quad \rhd_T \quad ([1, 2], c)$$

Printing

Output literals $o \in \mathsf{O}$.

$$T(X) \stackrel{\text{def}}{:=} \mathsf{O}^* \times X$$

$$c ::= \cdots \mid \mathsf{print}(o).\, c$$

$$\mathsf{print}(o).\, c \rhd_T ([o], c)$$

Repeated reductions, Kleisli composition :

$$\mathsf{print}(1).\, \mathsf{print}(2).\, c \qquad \rhd_T \qquad ([1], \mathsf{print}(2).\, c) \qquad \rhd_T \qquad ([1, 2], c)$$

Other effects :

- $\rightsquigarrow$ Non-termination ($TX = 1 + X, \mathsf{fail}$),
- $\rightsquigarrow$ Non-determinism ($TX = \mathcal{P}_{fin}(X), \mathsf{flip}()$).

Effects and polarity: as it should

“(print 1; (x, y))”:

$$\mu^{+}\alpha.\,\mathsf{print}(1).\,\langle (x, y) \parallel \alpha \rangle^{+}$$

Effects and polarity: as it should

"(print 1; (x, y))":

$$\mu^{+}\alpha.\,\mathsf{print}(1).\,\langle (x, y) \parallel \alpha \rangle^{+}$$

Positive term : not a value, will perform the effect first.

Effects and polarity: as it should

“(print 1; (x, y))”:

$$\mu^+\alpha.\,\mathsf{print}(1).\,\langle (x, y) \,\|\, \alpha\rangle^+$$

Positive term : not a value, will perform the effect first.

“(print 1; $\lambda x.\, x$)”:

$$\mu^-\alpha.\,\mathsf{print}(1).\,\langle \lambda x.\, x \,\|\, \alpha\rangle^-$$

Negative term : value, will block effects until the function is forced.

Effects and polarity: as it should

"(print 1; (x, y))":

$$\mu^{+}\alpha.\, \mathsf{print}(1).\, \langle (x, y) \,\|\, \alpha \rangle^{+}$$

Positive term : not a value, will perform the effect first.

"(print 1; $\lambda x.\, x$)":

$$\mu^{-}\alpha.\, \mathsf{print}(1).\, \langle \lambda x.\, x \,\|\, \alpha \rangle^{-}$$

Negative term : value, will block effects until the function is forced.

$$\begin{array}{rcl} (t : A \to B) & \rhd_{\eta} & \mu(x \cdot \alpha).\, \langle t \,\|\, x \cdot \alpha \rangle \\ (e : A \otimes B) & \rhd_{\eta} & \tilde{\mu}(x_1, x_2).\, \langle (x_1, x_2) \,\|\, e \rangle \end{array}$$

Effects and polarity: as it should

"(print 1; (x, y))":

$$\mu^{+}\alpha.\,\mathsf{print}(1).\,\langle (x, y) \parallel \alpha \rangle^{+}$$

Positive term : not a value, will perform the effect first.

"(print 1; $\lambda x.\, x$)":

$$\mu^{-}\alpha.\,\mathsf{print}(1).\,\langle \lambda x.\, x \parallel \alpha \rangle^{-}$$

Negative term : value, will block effects until the function is forced.

$$\begin{array}{rcl} (t : A \to B) & \rhd_{\eta} & \mu(x \cdot \alpha).\,\langle t \parallel x \cdot \alpha \rangle \\ (e : A \otimes B) & \rhd_{\eta} & \tilde{\mu}(x_1, x_2).\,\langle (x_1, x_2) \parallel e \rangle \end{array}$$

We already had the right η-laws to account for effects.

Purity and substitution

A pure term t can be substituted in any context (referential transparency).
What should be a pure coterm e ?

Purity and substitution

A pure term t can be substituted in any context (referential transparency).
What should be a pure coterm e ?

$\rightsquigarrow$ Linearity and thunkability

Symmetric notions of purity from **substitution properties**.

Thunkable terms

t **thunkable** ("t is pure") $\overset{\text{def}}{:=}$

$$\forall c[x], \quad c\,[t/x] \simeq \langle t \,\|\, \tilde{\mu}x.\, c \rangle$$

Thunkable terms

t **thunkable** ("t is pure") $\stackrel{\text{def}}{:=}$

$$\forall c[x], \quad c\,[t/x] \simeq \langle t \,\|\, \tilde{\mu}x.\, c \rangle$$

All V are thunkable.

Thunkable terms

t **thunkable** ("t is pure") $\stackrel{\text{def}}{:=}$

$$\forall c[x], \quad c\,[t/x] \simeq \langle t \,\|\, \tilde{\mu}x.\, c \rangle$$

All V are thunkable. Semantic notion of value.

Thunkable terms

t **thunkable** (“t is pure”) $\stackrel{\text{def}}{:=}$

$$\forall c[x], \quad c\,[t/x] \simeq \langle t \,\|\, \tilde{\mu}x.\, c \rangle$$

All V are thunkable. Semantic notion of value.

Not thunkable : $t \stackrel{\text{def}}{:=} \mu^{+}\alpha.\, \mathrm{print}(1).\, \langle x \,\|\, \alpha \rangle$.

$$c\,[t/x] \not\simeq \langle t \,\|\, \tilde{\mu}x.\, c \rangle$$

For example:

Thunkable terms

t **thunkable** ("t is pure") $\stackrel{\text{def}}{:=}$

$$\forall c[x], \quad c\,[t/x] \simeq \langle t \,\|\, \tilde{\mu}x.\, c\rangle$$

All V are thunkable. Semantic notion of value.

Not thunkable : $t \stackrel{\text{def}}{:=} \mu^{+}\alpha.\,\mathsf{print}(1).\,\langle x \,\|\, \alpha\rangle$.

$$c\,[t/x] \not\simeq \langle t \,\|\, \tilde{\mu}x.\, c\rangle$$

For example:

$\mathsf{print}(2).\,c'$ (non-commutative)

Thunkable terms

t **thunkable** ("t is pure") $\stackrel{\text{def}}{:=}$

$$\forall c[x], \quad c\,[t/x] \simeq \langle t \,\|\, \tilde{\mu}x.\, c\rangle$$

All V are thunkable. Semantic notion of value.

Not thunkable : $t \stackrel{\text{def}}{:=} \mu^{+}\alpha.\,\mathrm{print}(1).\,\langle x \,\|\, \alpha\rangle$.

$$c\,[t/x] \not\simeq \langle t \,\|\, \tilde{\mu}x.\, c\rangle$$

For example:

$$\Big\langle x \,\Big\|\, \tilde{\mu}x_1.\, \langle x \,\|\, \tilde{\mu}x_2.\, \rangle c' \Big\rangle \qquad \text{(duplicating)}$$

Thunkable terms

t **thunkable** ("t is pure") $\stackrel{\text{def}}{:=}$

$$\forall c[x], \quad c\,[t/x] \simeq \langle t \,\|\, \tilde{\mu}x.\, c \rangle$$

All V are thunkable. Semantic notion of value.

Not thunkable : $t \stackrel{\text{def}}{:=} \mu^{+}\alpha.\,\mathrm{print}(1).\,\langle x \,\|\, \alpha \rangle$.

$$c\,[t/x] \not\simeq \langle t \,\|\, \tilde{\mu}x.\, c \rangle$$

For example:

$$\langle y \,\|\, \star \rangle \qquad\qquad \text{(erasing)}$$

Thunkable terms

t **thunkable** ("t is pure") $\stackrel{\text{def}}{:=}$

$$\forall c[x], \quad c\,[t/x] \simeq \langle t \,\|\, \tilde{\mu}x.\, c \rangle$$

All V are thunkable. Semantic notion of value.

Not thunkable : $t \stackrel{\text{def}}{:=} \mu^{+}\alpha.\,\mathrm{print}(1).\,\langle x \,\|\, \alpha \rangle$.

$$c\,[t/x] \not\simeq \langle t \,\|\, \tilde{\mu}x.\, c \rangle$$

For example:

$$\langle y \,\|\, \star \rangle \qquad \text{(erasing)}$$

Thunkable : observationally pure commands

$$\mu^{+}\alpha.\,\langle V \,\|\, \alpha \rangle$$

$$\mu\alpha.\,\mathrm{flip}(x).\,\big\langle x \,\big\|\, \tilde{\mu}[(\sigma_1\,\_).\,\langle V \,\|\, \alpha \rangle \mid (\sigma_2\,\_).\,\langle V \,\|\, \alpha \rangle]\big\rangle$$

Linear co-terms

e **linear** ("e is strict") $\stackrel{\text{def}}{:=}$

$$\forall c[\alpha], \quad c\,[e/\alpha] \simeq \langle \mu\alpha.\, c \,\|\, e \rangle$$

All S are linear. Semantic notion of co-value.

Not linear : $e \stackrel{\text{def}}{:=} \tilde{\mu}^{-}x.\, \langle y \,\|\, \alpha \rangle$

$$\text{take } c[\alpha] \stackrel{\text{def}}{:=} \text{fail} \qquad \rightsquigarrow \qquad c\,[e/\alpha] \not\simeq \langle \mu\alpha.\, c \,\|\, e \rangle$$

Purity and polarity

Consider a one-one command

$$c[y, \beta] : (y : A \vdash \beta : B)$$

$$\begin{aligned} c[y,\beta] \text{ thunkable} \quad &\overset{\text{def}}{:=} \quad \mu\beta.\, c[y,\beta] \text{ thunkable.} \\ c[y,\beta] \text{ linear} \quad &\overset{\text{def}}{:=} \quad \tilde{\mu}y.\, c[y,\beta] \text{ linear.} \end{aligned}$$

Purity and polarity

Consider a one-one command

$$c[y,\beta] : (y : A \vdash \beta : B)$$

$$c[y,\beta] \text{ thunkable} \quad \stackrel{\text{def}}{:=} \quad \mu\beta.\, c[y,\beta] \text{ thunkable.}$$

$$c[y,\beta] \text{ linear} \quad \stackrel{\text{def}}{:=} \quad \tilde{\mu}y.\, c[y,\beta] \text{ linear.}$$

$$\mu\beta.\, c[y,\beta] \quad \text{thunkable}$$

$$\stackrel{\text{def}}{:=}$$

$$\forall c'[x], \qquad c'\,[\mu\beta.\, c[y,\beta]/x] \quad \simeq \quad \langle \mu\beta.\, c[y,\beta] \,\|\, \tilde{\mu}x.\, c'[x] \rangle^{\varepsilon}$$

Purity and polarity

Consider a one-one command

$$c[y, \beta] : (y : A \vdash \beta : B)$$

$$c[y,\beta] \text{ thunkable} \quad \overset{\text{def}}{:=} \quad \mu\beta.\, c[y,\beta] \text{ thunkable.}$$
$$c[y,\beta] \text{ linear} \quad \overset{\text{def}}{:=} \quad \tilde{\mu}y.\, c[y,\beta] \text{ linear.}$$

$$\mu\beta.\, c[y,\beta] \quad \text{thunkable}$$

$$\overset{\text{def}}{:=}$$

$$\forall c'[x], \qquad c'\,[\mu\beta.\, c[y,\beta]/x] \quad \simeq \quad \langle \mu\beta.\, c[y,\beta] \parallel \tilde{\mu}x.\, c'[x] \rangle^{\varepsilon}$$

All $y : A \vdash \beta : N$ are thunkable.

Purity and polarity

Consider a one-one command

$$c[y,\beta] : (y : A \vdash \beta : B)$$

$$\begin{aligned} c[y,\beta] \text{ thunkable} &\quad \stackrel{\text{def}}{:=} \quad \mu\beta.\, c[y,\beta] \text{ thunkable.} \\ c[y,\beta] \text{ linear} &\quad \stackrel{\text{def}}{:=} \quad \tilde{\mu}y.\, c[y,\beta] \text{ linear.} \end{aligned}$$

$$\mu\beta.\, c[y,\beta] \quad \text{thunkable}$$

$$\stackrel{\text{def}}{:=}$$

$$\forall c'[x], \qquad c'\,[\mu\beta.\, c[y,\beta]/x] \quad \simeq \quad \langle \mu\beta.\, c[y,\beta] \,\|\, \tilde{\mu}x.\, c'[x]\rangle^{\varepsilon}$$

All $y : A \vdash \beta : N$ are thunkable.
All $y : P \vdash \beta : B$ are linear.

Conclusion

L : polarized versions of the $\lambda\mu\tilde{\mu}$ abstract-machine calculi.

Future pointer: our upcoming tutorial!

Claim: a good **syntax** to talk about

- classical computation
- mixed-polarity equivalences
- effects
- purity

Thanks!

Andreas Abel, Brigitte Pientka, David Thibodeau, and Anton Setzer. Copatterns: Programming infinite structures by observations. In **POPL**, 2013. URL http://www.cse.chalmers.se/~abela/popl13.pdf.

Pierre-Louis Curien and Hugo Herbelin. The duality of computation. In **ICFP**, 2000. URL http://pauillac.inria.fr/~herbelin/publis/icfp-CuHer00-duality+errata.pdf.

Paul Downen and Zena Ariola. Tutorial on computational classical logic and the sequent calculus. In **JFP**, 2018. URL http://ix.cs.uoregon.edu/~pdownen/publications/sequent-intro.pdf.

Paul Downen, Philip Johnson-Freyd, and Zena M. Ariola. Structures for structural recursion. In **ICFP**, 2015. URL http://ix.cs.uoregon.edu/~pdownen/publications/structure-recursion-extended.pdf.

Étienne Miquey. A classical sequent calculus with dependent types. In **ESOP**, 2017.

Gabriel Scherer. Deciding equivalence with sums and the empty type. In **POPL**, 2017.

(Non-)Categorical models

One-one commands $P \vdash Q$ form a category.
One-one commands $M \vdash N$ form a category.

General one-one commands $A \vdash B$ do **not** form a category:
composition may be non-associative: $A \xrightarrow{f} P \xrightarrow{g} N \xrightarrow{h} B$

Adjunction(s) models

Two adjunctions?

- $\langle \_ \rangle^{-} \dashv \langle \_ \rangle^{+}$
- $\langle \_ \rangle^{+} \dashv \langle \_ \rangle^{-}$

Adjunction(s) models

Two adjunctions?

- $\langle \_ \rangle^- \dashv \langle \_ \rangle^+$
- $\langle \_ \rangle^+ \dashv \langle \_ \rangle^-$

$$\dfrac{\dfrac{\langle P \rangle^- \overset{\text{lin}}{\to} N}{P \overset{\text{lin+thunk}}{\to} N}}{P \overset{\text{thunk}}{\to} \langle N \rangle^-} \qquad \dfrac{\dfrac{\langle N \rangle^+ \to P}{N \to P}}{N \to \langle P \rangle^-}$$

The first is between pure commands (both linear and thunkable). Minimal effects.

Adjunction(s) models

Two adjunctions?

- $\langle \_ \rangle^- \dashv \langle \_ \rangle^+$
- $\langle \_ \rangle^+ \dashv \langle \_ \rangle^-$

$$\dfrac{\dfrac{\langle P \rangle^- \overset{\text{lin}}{\to} N}{P \overset{\text{lin+thunk}}{\to} N}}{P \overset{\text{thunk}}{\to} \langle N \rangle^-} \qquad \dfrac{\dfrac{\langle N \rangle^+ \to P}{N \to P}}{N \to \langle P \rangle^-}$$

The first is between pure commands (both linear and thunkable).
Minimal effects.

The second is between all commands.
Polarised effects.