Engineering logical relations for MLTT in Coq

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Martin-Löf Type Theory and its implementations

Martin-Löf logical framework + type formers $(\square, \Pi, \Sigma, x =_A y, \ldots)$

$$\Gamma \vdash \qquad \Gamma \vdash A \qquad \Gamma \vdash A \equiv B$$

$$\Gamma \vdash t : A \qquad \Gamma \vdash t \equiv u : A$$

Idealized metatheory of various proofs assistants:

Agda

Idris

LEAN
THEOREM PROVER

Practical implementation $\rightsquigarrow$ algorithms deciding each judgements

Formalized Metatheory of Type Theory: Why ?

This talk: A formalization of MLTT in Coq

Logical aspects

- Relative consistency
- Normalization/Canonicity
- Proof-theoretical bounds

Certification aspects

- Correctness, completeness and totality of the implemented algorithms

Formalized Metatheory of Type Theory: State of the art

Metacoq

Assumes normalization

Logical relations for MLTT

Decidability of Conversion for Type Theory in Type Theory

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Type theory should be able to handle its own meta-theory, both to justify its foundational claims and to obtain a verified implementation. At the core of a type checker for intensional type theory lies an algorithm to check equality of types, or in other words, to check whether two types are convertible. We have formalized in Agda a practical conversion checking algorithm for a dependent type theory with one universe à la Russell, natural numbers, and η-equality for Π types. We prove the algorithm correct via a Kripke logical relation parameterized by a suitable notion of equivalence of terms. We then instantiate the parameterized fundamental lemma twice: once to obtain canonicity and injectivity of type formers, and once again to prove the completeness of the algorithm. Our proof relies on inductive-recursive definitions, but not on the uniqueness of identity proofs. Thus, it is valid in variants of intensional Martin-Löf Type Theory as long as they support induction-recursion, for instance, Extensional, Observational, or Homotopy Type Theory.

CCS Concepts: • **Theory of computation** → **Type theory**; *Proof theory*.

Additional Key Words and Phrases: Dependent types, Logical relations, Formalization, Agda

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1 INTRODUCTION

A fundamental component of the implementation of a typed functional programming language is an algorithm that checks equality of types; even more so for dependently-typed languages where

Rely on Induction-Recursion

A Coq Formalization of Normalization by Evaluation for Martin-Löf Type Theory

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Abstract

We present a Coq formalization of the normalization-by-evaluation algorithm for Martin-Löf dependent type theory with one universe and judgmental equality. The end results of the formalization are certified implementations of a reduction-free normalizer and of a decision procedure for term equality.

The formalization takes advantage of a graph-based variant of the Bove-Capretta method to encode mutually recursive evaluation functions with nested recursive calls. The proof of completeness, which uses the PER-model of dependent types, is formalized by relying on impredicativity of the Coq system rather than on the commonly used induction-recursion scheme which is not available in Coq. The proof of soundness is formalized by encoding logical relations as partial functions.

CCS Concepts • **Theory of computation** → **Proof theory**; **Type theory**.

Keywords normalization by evaluation, type theory, Coq, program certification

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1 Introduction

Proof assistants such as Coq or Agda rely on constructive type theories, where logic is obtained through the Curry-Howard correspondence. In this paradigm propositions are identified with types, proofs are identified with terms, and proof validation is obtained by *type checking*. Thus, crucially, the type-checking in such systems has to be decidable. No less important is the *normalization* property, which states that each typable expression can be reduced to a normal form, since it plays a crucial role in a proof of the consistency of the logic. These two properties are intimately connected in the presence of *dependent types* i.e., types that depend on terms and that may themselves depend on the terms of the logic, which can be used to encode predicates. Since terms are present at the type level the type-checking algorithm must be equipped with a procedure for deciding equality over terms of the system. Furthermore, the presence of the universe – the type of types – requires one also to normalize types as part of the type-checking algorithm. Proving correctness of such rich type theories is hard and any seemingly innocent extensions can introduce issues that make it even harder. For instance, we may wish to equip the underlying type theory of a proof assistant with η-rules, since they allow us to equate terms like $P(f)$ and $P(\lambda x\,.\;A.\;f\,x)$, which turns out to be convenient in practice. However, such an extension may break the Church-Rosser [Takahashi 1995] property that is

Rely on Impredicativity

Main goal & Hauptsatz

Informally: Normalization of Coq in Coq

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Theorem :Typing and conversion are decidable for MLTT
wrt the theory of Coq .

MLTT with Π, Σ, $\mathbb{0}$, $\mathbb{1}$, $\mathbb{N}$, List

The theory of Coq: PCUIC

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Current gap, indexed inductive types and a hierarchy of universes.

Towards decidability

Declarative typing

- Free standing conversion rule

$$\frac{\Gamma \vdash_{\mathsf{de}} t : A \qquad \Gamma \vdash_{\mathsf{de}} A \cong B}{\Gamma \vdash_{\mathsf{de}} t : B}$$

- Conversion mixes arbitrary uses of congruence, computation (β), extensionality and transitivity steps.

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Algorithmic typing (bidirectional)

- Conversion constrained to phase changes

$$\frac{\Gamma \vdash_{\mathsf{al}} t \triangleright A \qquad \Gamma \vdash_{\mathsf{al}} A \cong B}{\Gamma \vdash_{\mathsf{al}} t \triangleleft B}$$

- Conversion guided by the terms: alternating weak-head reduction and syntax directed congruences/extensionality rules

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Bridging the gap between declarative and algorithmic

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Algorithmic $\to$ **Declarative:** Admissibility of algorithmic rules ✓

Declarative $\to$ **Algorithmic:** Need to show that every derivation has a canonical form

A logical relation for iterated whnf

A (proof-relevant) predicate

$$\Gamma \Vdash A$$

characterizing types by their
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For $[A] : \Gamma \Vdash A$, 3 predicates:

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Using **small-induction recursion** [Hancock et al.] in Coq.

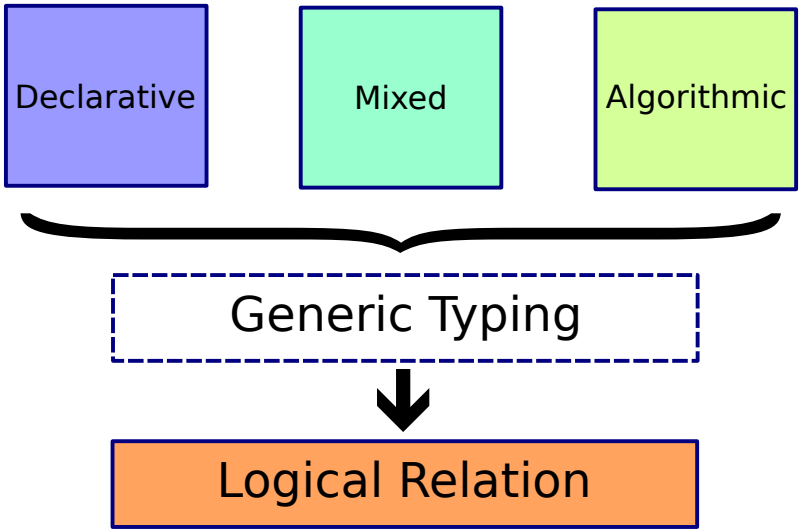
```
Inductive LR@{i j k} {l : TypeLevel} (rec : forall l', l' << l -> RedRel@{i j})
: RedRel@{j k} :=
  | LRU {Γ A} (H : [Γ ||-U<l> A]) :
      LR rec Γ A
      (fun B   => [Γ ||-U≅ B ])
      (fun t   => [ rec | Γ ||-U t     : A | H ])
      (fun t u => [ rec | Γ ||-U t ≅ u : A | H ])
  | LRne {Γ A} (neA : [ Γ ||-ne A ]) :
      LR rec Γ A
      (fun B   =>  [ Γ ||-ne A ≅ B     | neA])
      (fun t   =>  [ Γ ||-ne t     : A | neA])
      (fun t u =>  [ Γ ||-ne t ≅ u : A | neA])
  | LRPi {Γ : context} {A : term} (ΠA : PiRedTy@{j} Γ A) (ΠAad : PiRedTyAdequat
    LR rec Γ A
      (fun B   => [ Γ ||-Π A ≅ B     | ΠA ])
      (fun t   => [ Γ ||-Π t     : A | ΠA ])
      (fun t u => [ Γ ||-Π t ≅ u : A | ΠA ])
  | LRNat {Γ A} (NA : [Γ ||-Nat A]) :
    LR rec Γ A (NatRedTyEq NA) (NatRedTm NA) (NatRedTmEq NA)
  | LREmpty {Γ A} (NA : [Γ ||-Empty A]) :
    LR rec Γ A (EmptyRedTyEq NA) (EmptyRedTm NA) (EmptyRedTmEq NA)
  | LRSig {Γ : context} {A : term} (ΣA : SigRedTy@{j} Γ A) (ΣAad : SigRedTyAdeq
    LR rec Γ A (SigRedTyEq ΣA) (SigRedTm ΣA) (SigRedTmEq ΣA)
  | LRList {Γ : context} {A : term}
      (LA : ListRedTyPack@{j} Γ A) (LAAd : ListRedTyAdequate@{j k} (LR rec) LA)
    LR rec Γ A
      (ListRedTyEq@{j} Γ A LA)
      (ListRedTm@{j} Γ A LA)
      (ListRedTmEq@{j} Γ A LA).
```

Properties of the logical relation

- Irrelevance (including universe level)
- Equivalence: reflexivity, symmetry, transitivity
- Neutral reflection
- Closure by anti-reduction

Fundamental lemma: if $\Gamma \vdash_{\mathsf{de}} t : A$ then $[A] : \Gamma \Vdash A$ and $\Gamma \Vdash_{[A]} t : A$

3 logical relations in 1



Engineering aspects

Code: 20k loc (9k spec; 11k proofs)

The formalization rely on

- `autosubst2` for generating renaming, substitution and their lemmas
- `Equations`
- `partialfun` (T. Winterhalter) for defining and reasoning on the typechecking algorithm

Tactics:

- for discharging typing goals (`eauto` with typing lemmas)
- in order to dispatch the many forms of irrelevance
- for instantiating the logical relation with valid substitutions

Current limitations and future steps

The formalization currently cover

- Only one universe
- Few (co)inductives: $\mathbb{W}$, $\mathbb{Id}$, $\mathbb{M}$ wanted !
  Add a scheme for cumulative indexed-inductives ?
- And some performance issues to tackle.

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A sandbox for experimentations on type theories and their normalization