A Reasonably Gradual Dependent Type Theory

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Types'22, Nantes
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Prototyping with dependent types


```
Inductive vect (A : Type) : ℕ → Type :=
| nil : vect A 0
| _::_ {n : ℕ} (head : A) (tail : vect A n) : vect A (n+1)
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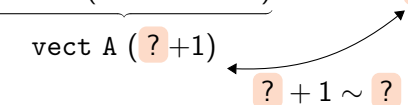
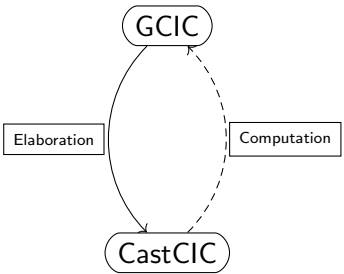


GCIC & CastCIC: a marriage of Gradual & Dependent Types



- Unknown terms and types $?_A : A$
- Casts $t : A \Longrightarrow \langle B \Leftarrow A \rangle\, t : B$
- Errors $\mathtt{err}_A : A$

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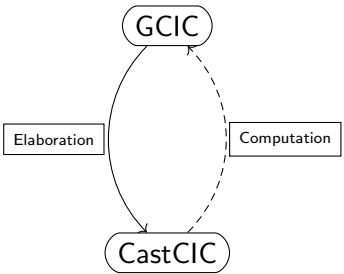

Precision between types

$\mathbb{N} \to \mathbb{N} \sqsubseteq \mathbb{N} \to ? \sqsubseteq ? \to ? \sqsubseteq ?$

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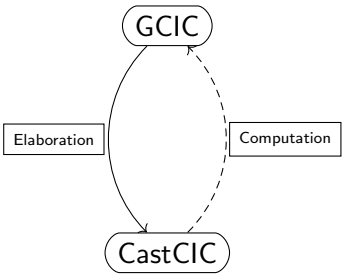

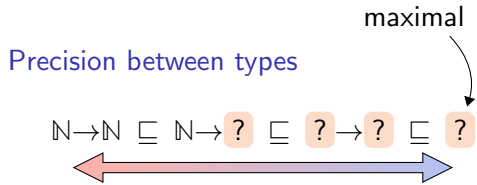

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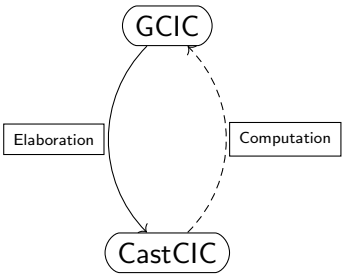

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maximal

Precision between types

$\mathbb{N} \to \mathbb{N} \sqsubseteq \mathbb{N} \to ? \sqsubseteq ? \to ? \sqsubseteq ?$

Precision enforces cast validity:

$$A \sqsubseteq B \implies A \underset{\langle A \Leftarrow B \rangle}{\overset{\langle B \Leftarrow A \rangle}{\rightleftarrows}} B$$

Graduality [New & Ahmed 2018]

The Fire Triangle of Graduality

Conservativity over CIC +
Gradual properties

$?_{\square} : \square$

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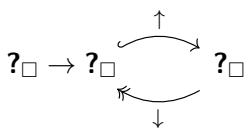

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$$?_\square \to ?_\square \quad \uparrow \downarrow \quad ?_\square$$

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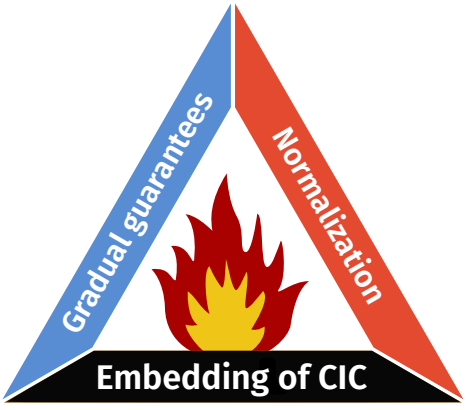

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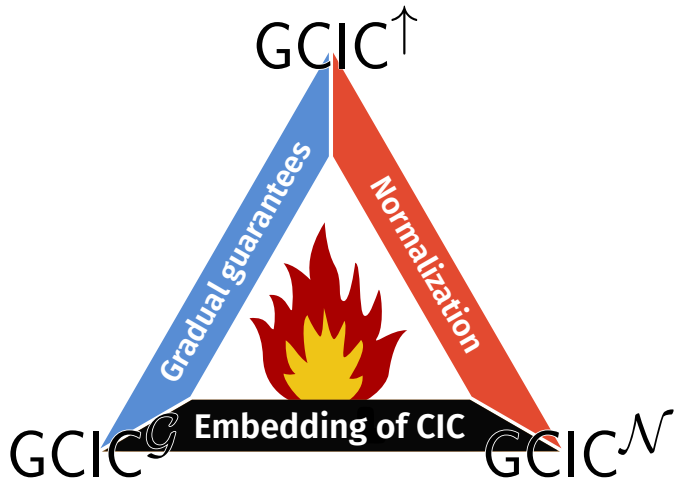

Internal precision in GRIP

$\mathsf{CastCIC}^{\mathcal{N}}$: Normalizing, conservative over CIC

How far is $\mathsf{CastCIC}^{\mathcal{N}}$ from being gradual ?

Internal precision in GRIP

$\mathsf{CastCIC}^{\mathcal{N}}$: Normalizing, conservative over CIC

$$\mathsf{GRIP} = \mathsf{CastCIC}^{\mathcal{N}} + \sqsubseteq$$

$$\frac{\Gamma \vdash A, B : \square_i}{\Gamma \vdash A \sqsubseteq_i B : \mathbb{P}} \qquad \frac{\Gamma \vdash A, B : \square_i \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash t \ {}_A\sqsubseteq_B \ u : \mathbb{P}}$$

$\mathbb{P}$: Pure universe of propositions

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$\mathbb{P}$: Pure universe of propositions

no error, no casts, no ?

definitionally proof-irrelevant

Proof of precision – An example

$$\texttt{add1} := \lambda x : \mathbb{N}.x + 1 \qquad \mathbb{N}\to\mathbb{N} \sqsubseteq_{\texttt{?}_\square \to \mathbb{N}} \qquad \texttt{add1?} := \lambda x : \texttt{?}_\square.(\langle \mathbb{N} \Leftarrow \texttt{?}_\square \rangle\, x) + 1$$

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- ▷ $\iota(?_{\square_i} \to ?_{\square_i}) \sqsubseteq_{i+1} ?_{\square_{i+1}}$

How good is GRIP?

Consistency $\nvdash_{\mathsf{GRIP}} t : \bot$ (where $\bot : \mathbb{P}$)

Idea: Build a model of GRIP in MLTT+SProp using partial pre-orders.
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Gradual fragment $\mathsf{GRIP}^{\uparrow}$

Idea: Restrict catch to match and raise universe levels on Π

$$\frac{\Gamma \vdash_{\mathsf{GRIP}^{\uparrow}} A : \square_i \qquad \Gamma, x : A \vdash_{\mathsf{GRIP}^{\uparrow}} B : \square_i}{\Gamma \vdash_{\mathsf{GRIP}^{\uparrow}} \Pi\, x : A.B : \square_{i+1}}$$

Summary

GRIP = partially gradual type theory + internal precision

- ▷ More details in the paper `https://hal.inria.fr/hal-03596652`
- ▷ Proof of concept for GRIP in Agda
- ▷ Partial pre-order based model in Coq

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GRIP is a type theory, not yet a proof assistant:

- Elaboration from a gradual source
- General inductive types
- An actual implementation ?

```
kyod@takasomen ~/W/G/G/GCIC (load-in-subdir)> esy repl
> def delta(x : ?T0) : ?T0 = (x : ?T0 -> ?T0) x;;
delta : Π(x : ?□0). ?□0 defined.
> def omega : ?T0 = delta (delta : ?T0);;
omega : ?□0 defined.
> eval omega;;
err_?_□0
> set variant G;;
OK
> eval omega;;
[reduction_error] not enough fuel
```

WIP GCIC prototype
(T. Díaz, S. Malewski, E. Tanter)